MUMT 618: Week #4

1 Wave Motion in Strings

We will first consider mechanical waves in solids. Wave motion in solids is generally a result of both elasticity and geometric stiffness. Elasticity refers to the ability of a deformed material to return to its original shape and size when the deforming load is removed. Geometric stiffness is the result of a applied pre-stress that causes the solid to depart from its equilibrium state. In the case of a one-dimensional stretched string, geometric stiffness dominates and we initially ignore elasticity (or bending stiffness).

1.1 The 1D Wave Equation

- The wave equation provides an analytic description of wave motion over time and through a spatial medium.
- This mathematical equation can subsequently be used to evaluate and model wave motion along the string and at boundaries.
- In analyzing the one-dimensional (1D) string section illustrated in Fig. 1 below, we make the following assumptions:
  - The mass per unit length of the string is constant and the string is perfectly elastic (there is no resistance to bending).
  - The tension caused by stretching the string before fixing it at its endpoints is so large that gravitational forces on the string are negligible.
  - The string moves only in the transverse direction and these deflections are small in magnitude.

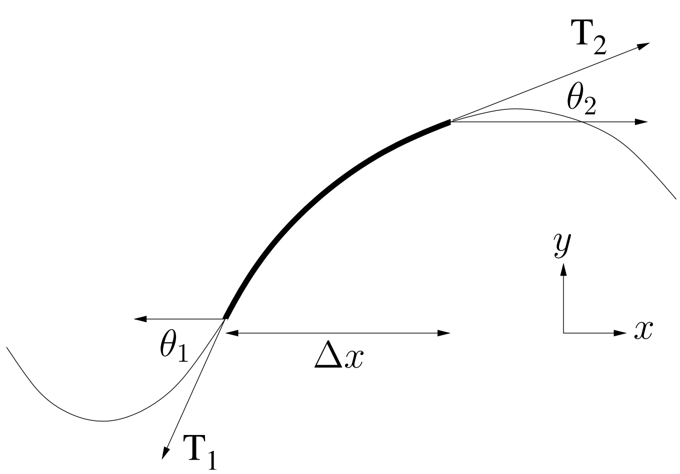

Figure 1: A short section of a stretched, deformed string.

- The mass of the short string section (length Δx) is $\epsilon\Delta x$, where ϵ is the mass per unit length of the string.
- Since there is no horizontal motion, the two horizontal components of tension must be constant: $T_1 \cos\theta_1 = T_2 \cos\theta_2 = T$.
- The net vertical force on the section is $T_2 \sin\theta_2 - T_1 \sin\theta_1$.
- By Newton's Second Law: $T_2 \sin\theta_2 - T_1 \sin\theta_1 = \epsilon\Delta x \frac{\partial^2 y}{\partial t^2}$.
- Making use of the horizontal tension components, we obtain:

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \tan\theta_2 - \tan\theta_1 = \frac{\epsilon\Delta x}{T}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

- The expressions $\tan\theta_2$ and $\tan\theta_1$ are the slopes of the string at the points x and $x + \Delta x$:

$$\tan\theta_1 = \left(\frac{\partial y}{\partial x}\right)_x \qquad \text{and} \qquad \tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}.$$

- Making these substitutions in Eq. 1 and dividing by Δx,

$$\frac{1}{\Delta x}\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \frac{\epsilon}{T}\frac{\partial^2 y}{\partial t^2}.$$

- Letting Δx approach zero, we obtain the linear partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\epsilon}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2},$$

  where $c = \sqrt{T/\epsilon}$ is the speed of wave motion on the string. This is the one-dimensional wave equation that describes small amplitude, lossless transverse waves on a stretched string.

1.2 Wave Equation Solutions

- A common approach to solving differential equations is to assume sinusoidal solutions in the form of complex exponentials.
- A time-harmonic representation of displacement can be expressed as $y(x,t) = \tilde{y}(x)e^{j\omega t}$, where ω is radian frequency and $\tilde{y}(x)$ represents the spatial distribution of the complex amplitude displacement.
- Substituting this expression into the wave equation yields the 1D form of the Helmholtz equation

$$\partial^2\tilde{y}/\partial x^2 + (\omega/c)^2\tilde{y}(x) = 0, \tag{2}$$

  which is a linear, second-order, ordinary differential equation (a function of x alone).
- A standard trial solution for Eq. (2) is of the form $\tilde{y}(x) = \tilde{A}e^{\lambda x}$, which after substitution yields:

$$(\lambda^2 + k^2)\tilde{y}(x) = 0,$$

  where λ is the wavelength and $k = \omega/c = 2\pi/\lambda$ is the wavenumber, which represents spatial frequency in radians per meter.
- There are two non-trivial solutions of $\lambda = \pm jk$, leading to a complete solution of the form:

$$y(x,t) = C^+ e^{j(\omega t - kx)} + C^- e^{j(\omega t + kx)},$$

  where C^+ and C^- are complex amplitudes.

- In terms of sinusoidal functions, this can expressed as:

$$\begin{aligned} y(x,t) &= A\sin\frac{\omega}{c}(ct-x) + B\cos\frac{\omega}{c}(ct-x) + C\sin\frac{\omega}{c}(ct+x) + D\cos\frac{\omega}{c}(ct+x) \\ &= A\sin(\omega t - kx) + B\cos(\omega t - kx) + C\sin(\omega t + kx) + D\cos(\omega t + kx). \end{aligned} \quad (3)$$

- This solution can be represented in a more general form, attributed to d'Alembert in 1747, of $y(x,t) = y^+(ct-x) + y^-(ct+x)$.
- $y^+(ct-x)$ represents a wave traveling in the positive x direction with a velocity c. Similarly, $y^-(ct+x)$ represents a wave traveling in the negative x direction with the same velocity. Each component is generally referred to as a *traveling wave*.
- The functions y^+ and y^- are arbitrary and of fixed shape (given our assumed lossless medium) ... see waves on string simulation.
- This implies that waves can propagate in two opposite directions in a one-dimensional medium.

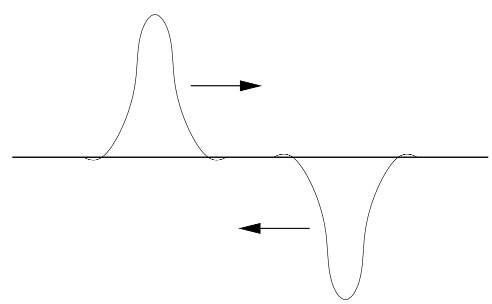

Figure 2: Traveling waves on a string.

- When two or more waves pass through the same region of space at the same time, the actual displacement is the vector (or algebraic) sum of the individual displacements (referred to as linear superposition).

1.3 Standing Waves

- Imagine a string of length L that is rigidly fixed at $x = 0$ and $x = L$. This implies boundary conditions of $y(0,t) = 0$ and $y(L,t) = 0$.
- Making use of $y(0,t) = 0$ and Eq. (3), we find that $A = -C$ and $B = -D$, so

$$y(x,t) = A[\sin(\omega t - kx) - \sin(\omega t + kx)] + B[\cos(\omega t - kx) - \cos(\omega t + kx)]. \quad (4)$$

- Using sinusoidal sum and difference formulae, $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$ and $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$, we obtain

$$\begin{aligned} y(x,t) &= 2A\sin(kx)\cos(\omega t) - 2B\sin(kx)\sin((\omega t) \\ &= 2[A\cos(\omega t) - B\sin(\omega t)]\sin(kx). \end{aligned}$$

- The boundary condition $y(L,t) = 0$ requires that $\sin(kL) = 0$ or $\omega L/c = n\pi$, so that ω is restricted to values $\omega_n = n\pi c/L$ or $f_n = nc/2L$.
- Thus, the rigidly fixed string has normal modes of vibration given by

$$y_n(x,t) = (A_n \sin(\omega_n t) + B_n \cos(\omega_n t))\sin(\omega_n x/c), \quad (5)$$

which are harmonic because each f_n is a integer multiple n times $f_1 = c/2L$. These correspond to the well known standing wave patterns on a stretched string.

2 Impedance

In this section, we are interested in understanding the concept of impedance, which relates the *frequency-dependent* nature of "force" and "motion" in a system. In subsequent sections of this course, we will evaluate the impedance of various parts of musical instruments to gain some understanding of how they vibrate.
An impedance provides a frequency response representation of a system in terms of input and output physical signals, in a way similar to the frequency response of filters. Likewise, the inverse frequency transform of an impedance provides an impulse response characterization.
Note that the input and output signals used to compute impedance can be taken at the same or different points of a system. If the input and output locations are at the input to a system, the resulting frequency-domain ratio is referred to as an *input* or *driving point impedance*. If the input and output locations are different, the frequency-domain ratio is referred to as a *transfer impedance*.

- In AC electrical systems, impedance is defined as voltage divided by current.
- Electrical *resistors* have constant, frequency-independent impedances.
- Electrical capacitors and inductors, however, have current-to-voltage characteristics that change with respect to the frequency of an applied source.
- For mechanical systems, impedance is defined as the ratio of force to velocity.
- The inverse of impedance is called *admittance*. One can use the term *immittance* to refer to either an impedance or an admittance.
- In lossless systems, an immittance is purely imaginary and called a *reactance*.
- Immittances are *steady state* characterizations that imply zero initial conditions for elements with "memory" (masses and springs, capacitors and inductances).
- In acoustics, impedance is given by pressure divided by either particle or volume velocity.
- More generally, we can derive (or measure) frequency-domain transfer function representations of vibrating systems in terms of impedances or admittances. As well, we can derive or calculate corresponding time-domain impulse responses from these characterizations.

2.1 Wave Impedance

- Consider a sinusoidal traveling-wave component of displacement moving in the positive x direction along an infinite string:
$$\tilde{y}^+(x,t) = Ae^{j(\omega t - kx)},$$
where A is a complex constant that describe the amplitude and phase of the traveling-wave component and $k = \omega/c = 2\pi/\lambda$.
- At an arbitrary point and time along the string (Fig. 3), the vertical force exerted on the portion to the right by the left-side portion of the string is given by
$$\tilde{f}^+(t,x) = -T\sin(\theta) \approx -T\tan(\theta) = -T\frac{\partial \tilde{y}^+}{\partial x} = TjkAe^{j(\omega t - kx)},$$
where T is the string tension and $|\partial \tilde{y}^+/\partial x| \ll 1$. This force is clearly balanced by an equal but opposite force exerted from the right-side portion of the string.
- The vertical velocity of the wave component $\tilde{y}^+(t,x)$ is
$$\tilde{v}^+(t,x) = \frac{\partial \tilde{y}^+}{\partial t} = j\omega Ae^{j(\omega t - kx)}.$$

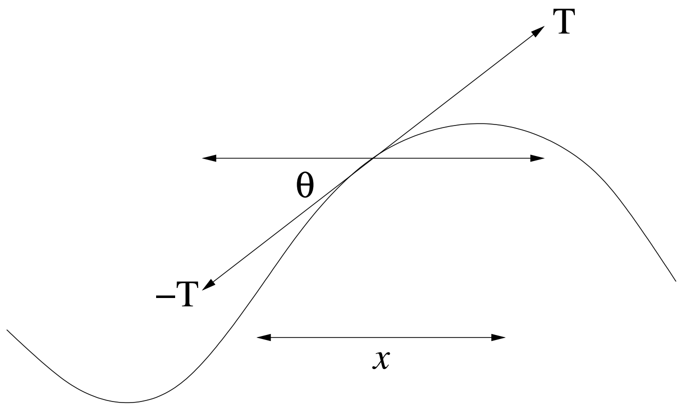

Figure 3: A short section of a stretched, vibrating string.

- The impedance of the wave component at this arbitrary point is the ratio of force to velocity, which is

$$\frac{\tilde{f}^+(t,x)}{\tilde{v}^+(t,x)} = T/c,$$

  where c is the speed of wave propagation. In deriving the wave equation for string motion, we determined $c = \sqrt{T/\epsilon}$ and thus T/c is also equal to ϵc, where ϵ is the linear mass density of the string.

- This fundamental scalar quantity is referred to as the *characteristic impedance* or *wave impedance* of the string and is denoted

$$R \triangleq \sqrt{T\epsilon} = \frac{T}{c} = \epsilon c.$$

  It indicates that the force and velocity components of a traveling wave moving along a string are related by a constant, frequency-independent "resistance".

- After similar consideration of a traveling-wave component moving to the left, a more general, frequency-independent expression for the relationship between vertical force and velocity on the string is:

$$f(t,x) = R\left[\frac{\partial}{\partial t}y^+(t-x/c) - \frac{\partial}{\partial t}y^-(t+x/c)\right] = R\left[v^+(t-x/c) - v^-(t+x/c)\right].$$

- Traveling-wave components of force are thus related to traveling-wave components of velocity by

$$\begin{aligned} f^+ &= Rv^+ \\ f^- &= -Rv^- \end{aligned}$$

2.2 The Driving Point Impedance of a Fixed String

- In systems with feedback, impedance (or admittance) provides information about resonances or anti-resonances.
- Consider a string of length L and tension T rigidly fixed at position $x = L$.
- Let us assume the string is being driven at position $x = 0$ by a transverse sinusoidal force in the form of a complex exponential: $\tilde{f}(t) = Fe^{j\omega t}$.

- The motion of a string of finite length will be composed of both right- and left-going wave components. The resulting sinusoidal response of the string can then be represented as:

$$\tilde{y}(x,t) = C^+ e^{j(\omega t - kx)} + C^- e^{j(\omega t + kx)}, \tag{6}$$

  where C^+ and C^- are complex constants that describe the amplitude and phase of each traveling-wave component with respect to the driving force and $k = \omega/c = 2\pi/\lambda$.

- At the fixed end ($x = L$), the boundary condition $y(L,t) = 0$ implies

$$0 = [C^+ e^{-jkL} + C^- e^{jkL}]e^{j\omega t}.$$

- At $x = 0$, the driving force must be compensated by a string force of $\tilde{F} \simeq -T(\partial \tilde{y}/\partial x)$, resulting in the expression

$$Fe^{j\omega t} = T(jkC^+ - jkC^-)e^{j\omega t}.$$

- These two equations can be used to solve for C^+ and C^-, which when substituted back into Eq. 6 gives

$$\tilde{y}(x,t) = \frac{F}{kT}\frac{\sin k(L-x)}{\cos kL}e^{j\omega t}.$$

- From this, the string velocity can be determined as:

$$\tilde{v}(x,t) = \frac{\partial \tilde{y}}{\partial t} = \frac{j\omega F}{kT}\frac{\sin k(L-x)}{\cos kL}e^{j\omega t}.$$

- The driving-point, or input, impedance $Z_{in}(\omega)$ is defined as the ratio of force to velocity at the driving point ($x = 0$):

$$Z_{in}(\omega) = \frac{\tilde{f}(t)}{\tilde{v}(0,t)} = \frac{-jkT}{\omega}\cot kL = -jR\cot kL,$$

  where R is the characteristic impedance of the string. This function is plotted below.

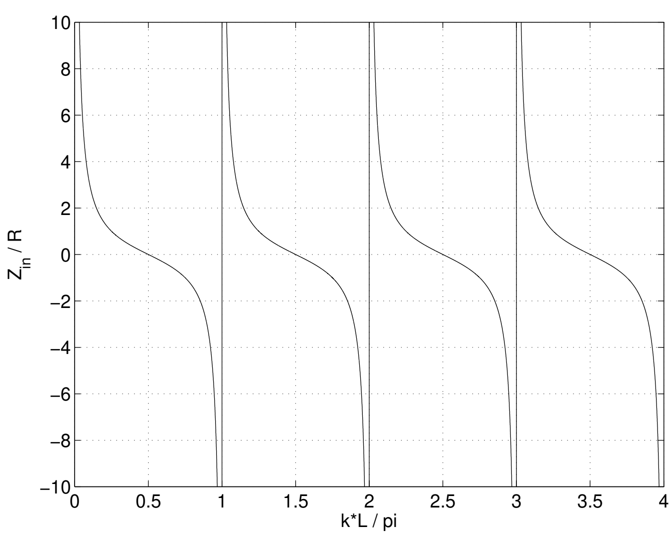

Figure 4: Normalized driving-point impedance of a string.

- The impedance is zero for values of $kL = \pi/2, 3\pi/2, \ldots$, and $\pm j\infty$ when $kL = 0, \pi, \ldots$ Note that this analysis does not account for any losses in the string or end support.

- If the driving point ($x = 0$) is fixed, the velocity of the string at that point must equal zero. This would correspond to an infinite impedance. Thus, the resonance frequencies of a string rigidly fixed at both its ends correspond to the frequencies at which $Z_{in} = \infty$, or when $kL = 0, \pi, \ldots$, from which we find the resonance frequencies $f_n = nc/(2L)$ for $n = 0, 1, 2, \ldots$ (which are the same as those found for the standing waves).

- On the other hand, if the point ($x = 0$) is perfectly free, the force of the string at that point must equal zero. This would correspond to an impedance of zero. Thus, the resonance frequencies of a string free on one end and rigidly fixed at the other correspond to the frequencies at which $Z_{in} = 0$, or when $kL = \pi/2, 3\pi/2, \ldots$, from which we find the resonance frequencies $f_n = (n+1)c/(4L)$ for $n = 0, 1, 2, \ldots$ (which occur at odd integer multiples of the fundamental $f_0 = c/(4L)$).

3 Digital Waveguide Theory

3.1 Sampled Traveling Waves

- The lossless one-dimensional wave equation was previously derived for a stretched string as:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where $c = \sqrt{T/\epsilon}$ is the speed of wave motion on the string, T is the string tension, and ϵ is the mass density of the string.

- The traveling-wave solution of the wave equation was published by d'Alembert in 1747. It has the general form

$$y(t, x) = y^+(t - x/c) + y^-(t + x/c),$$

for arbitrary functions $y^+(\cdot)$ and $y^-(\cdot)$. A function of $(t - x/c)$ can be interpreted as a fixed waveshape traveling to the right (positive x direction) and a function $(t + x/c)$ can be interpreted as a fixed waveshape traveling to the left (negative x direction), both with speed c.

- To develop a discrete-time model or simulation of traveling wave motion, it is necessary to sample the traveling-wave amplitudes in both time and space.

- The temporal sampling interval is T_s seconds, which corresponds to a sample rate $f_s \triangleq 1/T_s$ samples per second.

- The spatial sampling interval is given most naturally by $X \triangleq cT_s$ meters, or the distance traveled by sound in one temporal sampling interval. In this way, each traveling-wave component moves left or right one spatial sample for each time sample.

- Using the change of variables

$$\begin{aligned} x &\rightarrow x_m = mX \\ t &\rightarrow t_n = nT_s, \end{aligned}$$

the traveling-wave solution becomes

$$\begin{aligned} y(t_n, x_m) &= y^+(t_n - x_m/c) + y^-(t_n + x_m/c) \\ &= y^+(nT_s - mX/c) + y^-(nT_s + mX/c) \\ &= y^+[(n-m)T_s] + y^-[(n+m)T_s]. \end{aligned}$$

- This representation can be further simplified by suppressing T_s, with a resulting expression for physical displacement at time n and location m given as the sum of the two traveling-wave components

$$y(t_n, x_m) = y^+(n - m) + y^-(n + m). \tag{7}$$

3.2 Digital Waveguide Models

- The term $y^+[(n-m)T_s] = y^+(n-m)$ can be interpreted as the output of an m-sample delay line of input $y^+(n)$. Similarly, the term $y^-[(n+m)T_s] = y^-(n+m)$ can be interpreted as the input to an m-sample delay line with output $y^-(n)$.
- Physical wave variables are given by the superposition of traveling waves. In a one-dimensional system, we can use two systems of unit delay elements to model left- and right-going traveling waves and sum delay-line values at corresponding "spatial" locations to obtain physical outputs, as depicted below.

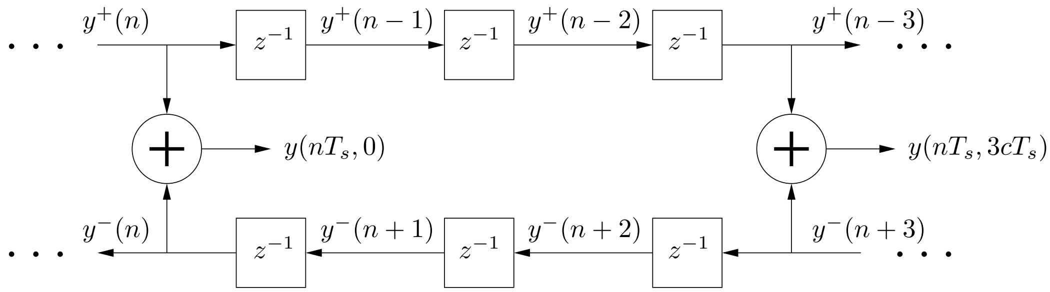

Figure 5: Discrete-time simulation of ideal, lossless wave propagation with observation points at $x = 0$ and $x = 3X = 3cT_s$.

- Any ideal, lossless, one-dimensional waveguide can be simulated in this way. The model is exact at the sampling instants to within the numerical precision of the processing system.
- To avoid aliasing, the traveling waveshapes must be initially bandlimited to less than half the sampling frequency.
- In many modeling contexts, the calculation of physical output values can be limited to just one or two discrete spatial locations. Individual unit delay elements are more typically combined and represented by digital delay lines, as shown below.

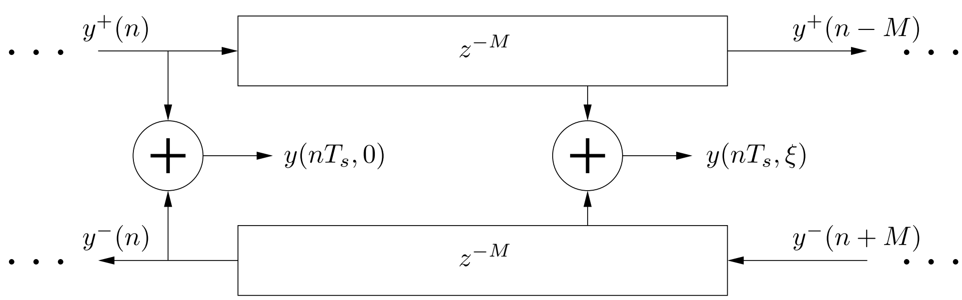

Figure 6: Digital waveguide simulation of ideal, lossless wave propagation using delay lines.

- The delay lines can be initialized with displacement data corresponding to any bandlimited, arbitrary waveshape.

3.3 Lossy Wave Propagation

- Real wave propagation is never lossless. Sound waves in air lose energy via molecular frictional forces. Mechanical vibrations in strings are dissipated through yielding terminations, the viscosity of the surrounding air, and via internal frictional forces. In general, these losses vary with frequency.

- Losses are often well approximated by the addition of a small number of terms to the wave equation.
- In the simplest case, we can add a frequency-independent force term that is proportional to the transverse string velocity. Using the wave equation derived for the string,

$$T\frac{\partial^2 y}{\partial x^2} = \epsilon\frac{\partial^2 y}{\partial t^2} + \mu\frac{\partial y}{\partial t},$$

where μ is the resistive proportionality constant. Assuming the friction coefficient is relatively small, the following general class of solutions to this equation can be found:

$$y(t, x) = e^{-(\mu/2\epsilon)x/c}y^+(t - x/c) + e^{(\mu/2\epsilon)x/c}y^-(t + x/c).$$

When this solution is sampled, we get

$$y(t_n, x_m) = g^m y^+(n - m) + g^{-m}y^-(n + m),$$

where $g = e^{-\mu T_s/2\epsilon}$.

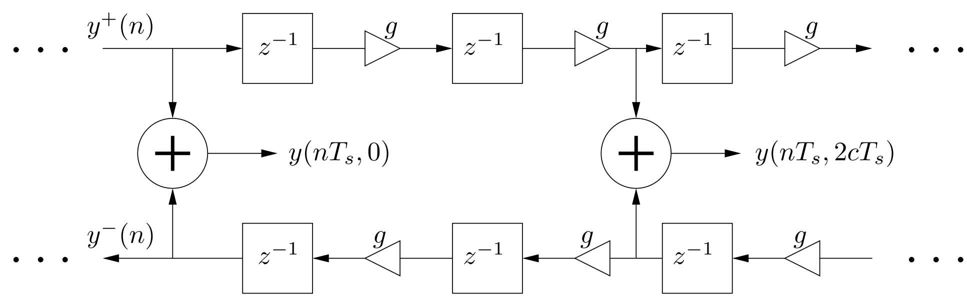

Figure 7: Discrete-time simulation of ideal, lossy wave propagation.

- Because the system is linear and time-invariant, the loss terms can be commuted and implemented at discrete points for efficiency.
- In the more realistic situation where losses are frequency dependent (and typically of "lowpass" characteristic), the g factors are replaced with frequency responses of the form $G(\omega)$. These responses can likewise be commuted and implemented at discrete spatial locations within the system.

3.4 Wave Scattering

- When a traveling-wave component experiences a change in wave impedance, it will be partly reflected from and partly transmitted through the impedance discontinuity in such a way that energy is conserved. This is referred to as *wave scattering*.

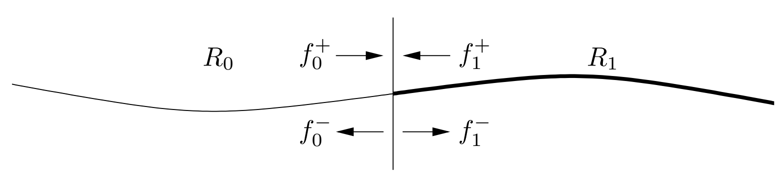

Figure 8: A string density discontinuity and associated traveling-wave scattering.

- Figure 8 depicts a string density discontinuity and associated traveling force wave components on each side of the junction.

- At the junction, we must have continuity of the medium, or there must be a common transverse string velocity at that point. Similarly, the forces on each side of the junction must be balanced.

- These constraints can be written

$$\begin{aligned} f_0 + f_1 &= 0 \quad \text{(forces on a massless point must balance)} \\ v_0 &= v_1 \quad \text{(string does not break)} \end{aligned}$$

- Let v denote the common transverse velocity of the string at the junction. Then the traveling-wave components of velocity can be written $v_i = v_i^+ + v_i^-$ or $v_i^- = v - v_i^+$.

- We can then write

$$\begin{aligned} 0 &= f_0 + f_1 = \left(f_0^+ + f_0^-\right) + \left(f_1^+ + f_1^-\right) \\ &= R_0\left(v_0^+ - v_0^-\right) + R_1\left(v_1^+ - v_1^-\right) = R_0\left(2v_0^+ - v\right) + R_1\left(2v_1^+ - v\right), \end{aligned}$$

  which can be solved for v as

$$v = 2\frac{R_0 v_0^+ + R_1 v_1^+}{R_0 + R_1}.$$

- Solving these expressions in terms of the outgoing traveling-wave components,

$$\begin{aligned} v_0^- &= v - v_0^+ = -\frac{R_1 - R_0}{R_1 + R_0} v_0^+ + \frac{2R_1}{R_1 + R_0} v_1^+ \\ v_1^- &= v - v_1^+ = \frac{2R_0}{R_1 + R_0} v_0^+ + \frac{R_1 - R_0}{R_1 + R_0} v_1^+. \end{aligned}$$

- It is standard to define the *reflection coefficient*

$$\rho \triangleq \frac{R_1 - R_0}{R_1 + R_0},$$

  which then allows us to write

$$\begin{aligned} v_0^- &= -\rho v_0^+ + (1+\rho) v_1^+ \\ v_1^- &= (1-\rho) v_0^+ + \rho v_1^+. \end{aligned}$$

- Figure 9 provides a block diagram implementation of the equations above. This is equivalent to a Kelly-Lochbaum junction representation [Kelly and Lochbaum, 1962].

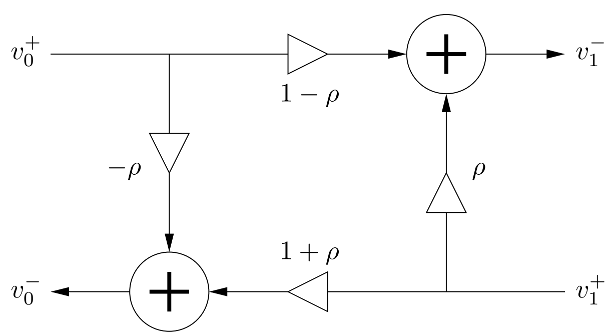

Figure 9: A traveling-wave scattering junction block diagram.

- The scattering equations can also be written

$$\begin{aligned} v_0^- &= v_1^+ + \rho(v_1^+ - v_0^+) \\ v_1^- &= v_0^+ + \rho(v_1^+ - v_0^+), \end{aligned}$$

  which requires only one multiplication and three additions (as diagrammed in Fig. 10).

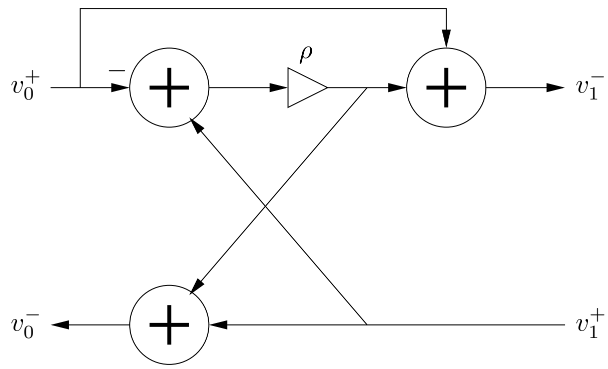

Figure 10: A one-multiply traveling-wave scattering junction block diagram.

3.5 Boundary Conditions

- Thus far, we have only considered wave propagation along or within a one-dimensional medium of seemingly infinite length. In an anechoic or non-reflecting waveguide, waves traveling in only one directon may exist and can thus be simulated with just a single delay line.
- In most situations, however, the media in which waves travel are of finite length and reflections occur at the boundaries that give rise to waves traveling in two directions per dimension.
- The simplest case is an ideal termination that is completely rigid.
- If we consider a string to be fixed at a position L, the boundary condition at that point is $y(t, L) = 0$ for all time. From the traveling-wave solution to the wave equation, we then have $y^+(t - L/c) = -y^-(t + L/c)$, which indicates that displacement traveling waves reflect from a fixed end with an inversion (or a *reflection coefficient* of -1).
- The simulation of ideal displacement wave motion in a string rigidly terminated at both its ends (and without losses) is shown in the figure below.

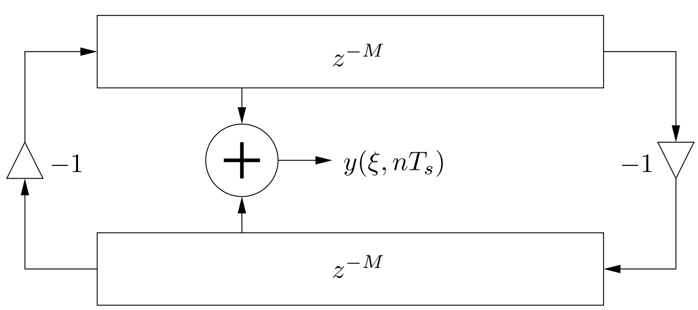

Figure 11: Digital waveguide simulation of wave propagation on a string fixed at both ends.

References

J. L. Kelly, Jr. and C. C. Lochbaum. Speech synthesis. In *Proceedings of the Fourth International Congress on Acoustics*, pages 1–4, Copenhagen, Denmark, Sept. 1962. Paper G42.