

MUMT 618: Week #5

1 String Instrument Acoustics

As we will see in this and subsequent sections, most musical instruments can be analyzed in terms of three principle acoustic components: a waveguide, an output filtering system, and an energy input mechanism. The corresponding elements of a general string instrument are its strings, body, and plucking, striking, or bowing techniques, respectively.

1.1 The String as a Waveguide

- The uniform stretched string forms a one-dimensional waveguide along which mechanical waves can travel with a constant speed of propagation (ignoring for the moment other dimensions of wave propagation).
- The time evolution of the string's shape is given by the linear superposition of traveling-wave components propagating in opposite directions along the string (Matlab example).
- The resonance frequencies (or normal modes) of the string are determined by its length, the wave speed of propagation, and the boundary conditions at each of its ends.
- For a string of length L , fixed at each of its ends, a discrete set of standing-wave patterns is possible with frequencies given by integer multiples of $f_0 = c/(2L)$, where c is the speed of wave propagation on the string. These standing-wave frequencies correspond to the resonances of the system.
- The extent to which the resonance frequencies of the string are present in any particular string vibration pattern is determined by the way the system is driven or excited.
- The stiffness of the string produces a restoring force that was neglected in our derivation of the wave equation. When we account for this force, the string modes are "stretched" from perfect harmonic relationships (with greater stretching for higher modes).
- For thin metal strings, the decay time is determined mostly by air viscosity. For gut or nylon strings, internal damping is dominant for most modes.

1.2 String Energy Input Mechanisms

1. The Plucked String:

- Plucking a string provides it with an initial energy displacement (potential energy).
- The shape of the string before its release completely defines the harmonic signature of the resulting motion.
- A string plucked at $1/n$ th the distance from one end will not have energy at the n th partial and its integer multiples.
- The strength of excitation of the n th vibrational mode is inversely proportional to the square of the mode number.

2. The Struck String:

- A struck string is given an initial velocity distribution (kinetic energy).
- A string struck at $1/n$ th the distance from one end will not have energy at the n th partial and its integer multiples.
- The harmonic amplitudes in the vibration spectrum of a struck string fall off less rapidly with frequency than those of plucked strings.
- Light “hammers” (mass much less than the mass of the string) result in little spectral drop-off with frequency. Heavier hammers produce a drop-off roughly proportional to the inverse of the mode number.

3. The Bowed String:

- “Stick-slip” mechanism: During the greater part of each vibration, the string is “stuck” to the bow and is carried with it in its motion. Then the string suddenly detaches itself and moves rapidly backward until it is caught again by the moving bow.
- Beginning and end of the slipping are triggered by the arrival of the propagating bend or “kink”.
- The string’s vertical motion at any one point is given by a sawtooth pattern.
- Round trip time depends only on the string length and the wave velocity.
- Bowing near the string end requires greater force and produces a louder, brighter sound than bowing farther from the end.
- Amplitude of vibration can be increased either by increasing the bow speed or by bowing closer to the bridge.

1.3 The Violin Body

1. Air and Wood Modes:

- For a violin, there are typically three or four important body resonances below 1 kHz. These include the first air mode, the $T1$ top plate mode, and the third and fourth “corpus” modes.
- Above 1 kHz, the mode structure is usually difficult to decipher, though a concentration of resonances around 2 to 3 kHz may be important (related to perception).

2. Tuning Top and Back Plates:

- Various techniques exist to measure the modal frequencies of the various components (Chladni patterns, force hammers, hologram interferometry).
- Complete systems, however, will have different mode structures than the individual plates.

3. The Bridge:

- The bridge transforms the motion of the vibrating string into a driving force on the top plate of the instrument.
- The violin bridge typically has strong resonances around 3000 and 6000 Hz.
- The bridge must move a small amount in order to transfer energy from the string to the body. For most musical instruments, however, the rate of energy transfer from the string to the bridge and soundboard is quite small (energy decay in the string is most affected by air viscosity and internal string damping).

1.4 Other Stringed Instruments

1. Guitars:

- At low frequencies, sound radiation occurs from both plates. At high frequencies, however, most of the radiated sound comes from the top plate.

2. Electric Guitars:

- The body has less effect on the sound. Electrical circuits replace the body filtering function.

2 String Modeling

In this section, we investigate digital waveguide methods for modeling simple string instruments. We also look at ways of modeling the way traveling-wave components propagate “through” impedance discontinuities.

2.1 The Ideal Plucked String

- If we consider a string to be fixed at a position L , the boundary condition at that point is $y(t, L) = 0$ for all time. From the traveling-wave solution to the wave equation, we then have $y_r(t-L/c) = -y_l(t+L/c)$, which indicates that displacement traveling waves reflect from a fixed end with an inversion (or a *reflection coefficient* of -1). The simulation of displacement wave motion in a string rigidly terminated at both its ends (and without losses) is shown in the figure below.

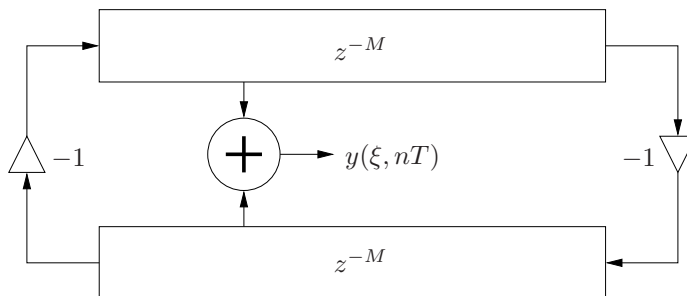


Figure 1: Digital waveguide simulation of ideal lossless wave propagation on a string fixed at both ends.

- An ideal plucked string is defined as having an initial displacement and zero initial velocity. In the model, the delay lines should be initialized with displacement data corresponding to some arbitrary initial string shape.
- Because the physical displacement of a string is given by the superposition of left- and right-going traveling waves, the initial amplitude of each delay-line section should be half the amplitude of the initial, physical string displacement.
- The initial displacement shape must be bandlimited to have the discrete-time sample rate. Because sharp corners imply an infinite bandwidth, plucking points should be rounded to some extent. In fact, this is physical given the stiffness of real strings and the finite size of plectra.
- It is possible to use digital waveguide models to simulate other wave variables as well, such as velocity or acceleration waves. Note that an ideal pluck shape corresponds to an acceleration impulse.
- For the moment, we ignore the fact that the string vibrations are influenced and filtered by the body resonances before being transmitted into the air.

2.2 The Ideal Struck String

- An ideal struck string involves zero initial displacement and a nonzero initial velocity distribution.
- For the struck string, simulation of velocity waves is possible. Alternately, the initial velocity distribution could be integrated with respect to x from $x = 0$, divided by c , and negated in the upper rail to obtain the equivalent initial displacement (since $v^\pm = \pm f^\pm/R = \mp cy'^\pm$, where R is the wave impedance of the string).

2.3 Rigid Terminations

- As discussed above, the boundary condition at a rigid termination for string displacement is $y(t, L) = 0$, which implies a reflection coefficient of -1.
- Since the displacement at a rigid termination is always zero, the physical velocity must also be zero for all time. Therefore, traveling-wave components of velocity will also reflect with a sign inversion at such a boundary.
- Remembering how the wave impedance relates traveling-wave components of force and velocity ($f^+ = Rv^+$, $f^- = -Rv^-$), as well as the reflection coefficient previously derived for velocity traveling waves, the condition for force wave components at a rigid termination is

$$f^+ = Rv^+ = -Rv^- = f^-.$$

Thus, traveling-wave components of force reflect at a rigid termination without a sign inversion (a reflection coefficient of +1).

2.4 The Damped Plucked String

- Ideal, lossless wave propagation does not occur in nature. As a first approximation to real propagation losses, we can make the substitution

$$z^{-1} \leftarrow gz^{-1}, \quad |g| \leq 1$$

in each delay line to obtain exponentially decaying traveling-wave propagation.

- Because the system is linear and time-invariant, these distributed loss factors (g) can be commuted or lumped together and implemented at discrete points. For a delay line of length N , the commuted factor would equal g^N .

2.5 Frequency-Dependent Damping & Stiffness

- A more accurate representation of propagation loss should be frequency-dependent. In general, damping will increase with frequency.
- To implement frequency-dependent loss, each scale factor g of the previous section must be replaced by an appropriately designed digital filter. Because the system is still linear and time-invariant, these filter responses can also be commuted and implemented as a single string loop filter $\hat{G}(z)$.
- In order to maintain stability (passivity), $\left| \hat{G}(e^{j\omega T}) \right| \leq 1$.
- A string fixed at both ends can be implemented with two delay lines, a discrete-time filter $\hat{G}(z)$ representing losses and inversion at the bridge, and an inversion representing reflection from the nut, as shown below. Additional simplifications are possible.
- String stiffness produces dispersion, or a variation of wave propagation speed with frequency. The use of an allpass filter in the string loop is effective in accurately simulating such dispersion.

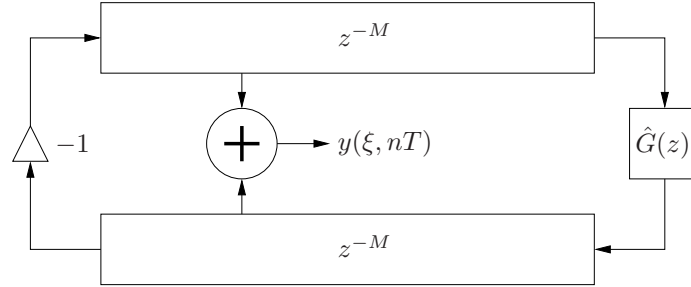
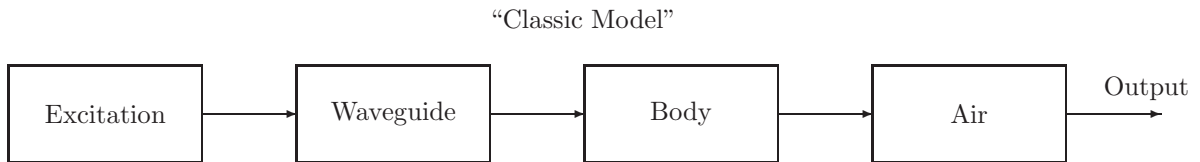


Figure 2: Digital waveguide simulation of dispersive wave propagation on a string fixed at both ends.

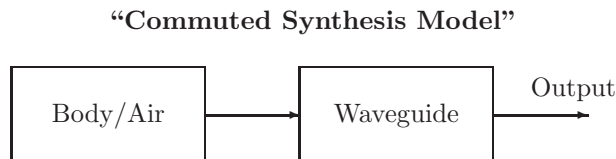
2.6 Commuted Waveguide Synthesis

Plucked string instruments are well modeled as linear systems and this allows great flexibility in the way they are implemented. Commuted synthesis takes advantage of this fact and provides a highly efficient way of producing high quality plucked-string sounds.

- The model components discussed thus far only account for vibrations of the string. In order to convincingly simulate a complete string instrument system, we must also account for the bridge and body response, as well as radiation patterns into the surrounding environment.
- Assuming we could develop representative filters for the various components of a string instrument, the output response of the complete system would be diagrammed and computed as shown below:



- For plucked or ideally struck strings, each component of the system is linear and time-invariant. Therefore, the system components can be rearranged without affecting the overall output.
- The body response of a string instrument is complex and generally requires a high-order digital filter to accurately simulate. However, by taking advantage of system commutativity, it is possible to record a body “impulse response” and use it as the string excitation. This technique is referred to as *commuted waveguide synthesis*. As shown, it is also possible to commute the radiation directivity response.



- The Matlab script `pluck.m` demonstrates committed waveguide synthesis, as well as the Karplus-Strong algorithm.

2.7 Commuted Synthesis Extensions

- From string acoustics, we know that a string plucked at $1/n$ th the distance from one end will not have energy at the n th partial and its integer multiples.
- If we initialize the delay lines of a waveguide model with a particular plucked string shape, the pluck position “filtering” is automatically included (it’s defined by the initial energy profile that is defined by the initial shape).

- In practice, however, the delay lines are either initialized with noise or driven with an external input signal (in which case they are initialized with zeros).
- For commuted synthesis, it is possible to pre-filter the input signal with a given pluck position filter. To maintain synthesis parameter flexibility, this is typically done during the synthesis computation.
- Pluck position filtering can be simulated with a feedforward comb filter, which creates notches in the frequency magnitude response at equal frequency intervals.

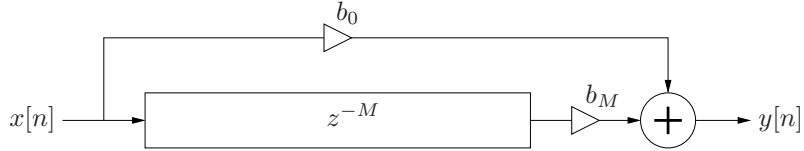


Figure 3: A feedforward comb filter block diagram.

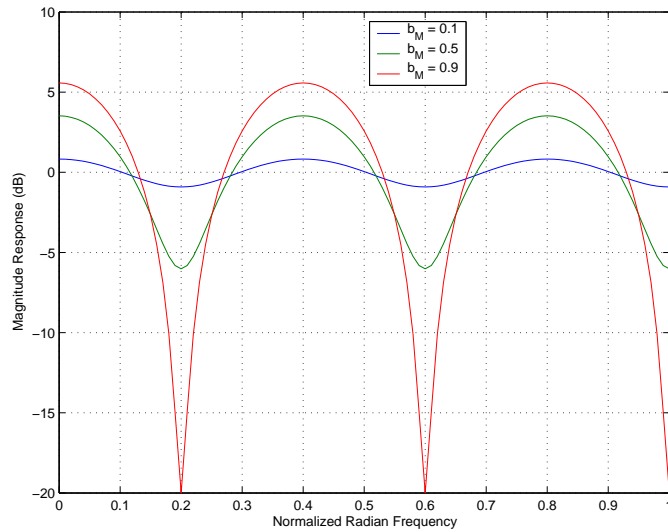


Figure 4: Magnitude response of a feedforward comb filter with $M = 5$, $b_0 = 1$, and $g = b_M = 0.1, 0.5$, and 0.9 .

- For $b_M > 0$, there are M peaks in the frequency response, centered about the frequencies $\omega_k = 2\pi k/M, k = 0, 1, \dots, M - 1$. Between these peaks, there are M notches at intervals of f_s/M Hz.
- When using commuted synthesis, body size can be roughly simulated with the body response playback rate, which shifts the body resonances higher or lower in frequency. This technique is demonstrated in the STK `Mandolin` class, implements commuted synthesis of a two string mandolin instrument.

2.8 Wave Scattering

- When a traveling-wave component experiences a change in wave impedance, it will be partly reflected from and partly transmitted through the impedance discontinuity in such a way that energy is conserved. This is referred to as *wave scattering*.
- Figure 5 depicts a string density discontinuity and associated traveling force wave components on each side of the junction.

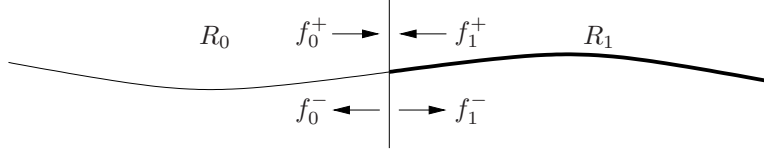


Figure 5: A string density discontinuity and associated traveling-wave scattering.

- At the junction, we must have continuity of the medium, or there must be a common transverse string velocity at that point. Similarly, the forces on each side of the junction must be balanced.
- These constraints can be written

$$\begin{aligned} f_0 + f_1 &= 0 & (\text{forces on a massless point must balance}) \\ v_0 &= v_1 & (\text{string does not break}) \end{aligned}$$

- Let v denote the common transverse velocity of the string at the junction. Then the traveling-wave components of velocity can be written $v_i = v_i^+ + v_i^-$ or $v_i^- = v - v_i^+$.
- We can then write

$$\begin{aligned} 0 &= f_0 + f_1 = (f_0^+ + f_0^-) + (f_1^+ + f_1^-) \\ &= R_0 (v_0^+ - v_0^-) + R_1 (v_1^+ - v_1^-) = R_0 (2v_0^+ - v) + R_1 (2v_1^+ - v), \end{aligned}$$

which implies

$$v = 2 \frac{R_0 v_0^+ + R_1 v_1^+}{R_0 + R_1}.$$

- Solving these expressions in terms of the outgoing traveling-wave components,

$$\begin{aligned} v_0^- &= v - v_0^+ = -\frac{R_1 - R_0}{R_1 + R_0} v_0^+ + \frac{2R_1}{R_1 + R_0} v_1^+ \\ v_1^- &= v - v_1^+ = \frac{2R_0}{R_1 + R_0} v_0^+ + \frac{R_1 - R_0}{R_1 + R_0} v_1^+. \end{aligned}$$

- It is standard to define the *reflection coefficient*

$$\rho \triangleq \frac{R_1 - R_0}{R_1 + R_0},$$

which then allows us to write

$$\begin{aligned} v_0^- &= -\rho v_0^+ + (1 + \rho) v_1^+ \\ v_1^- &= (1 - \rho) v_0^+ + \rho v_1^+. \end{aligned}$$

- Figure 6 provides a block diagram implementation of the equations above. This is equivalent to a Kelly-Lochbaum junction representation.
- The scattering equations can also be written

$$\begin{aligned} v_0^- &= v_1^+ + \rho(v_1^+ - v_0^+) \\ v_1^- &= v_0^+ + \rho(v_1^+ - v_0^+), \end{aligned}$$

which requires only one multiplication and three additions (as diagrammed in Fig. 7).

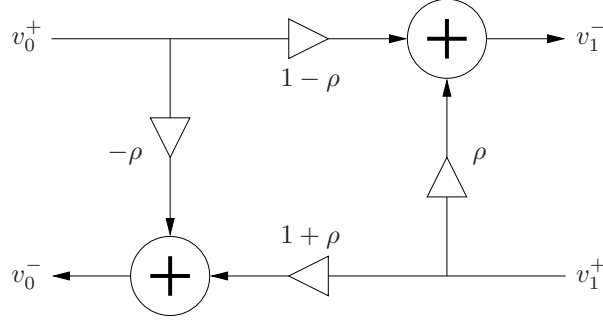


Figure 6: A traveling-wave scattering junction block diagram.

2.9 Coupled Strings

- Figure 8 depicts the case of two strings that terminate at a common, non-rigid bridge of impedance $R_b(\omega)$.
- The constraints at the junction are:

$$\begin{aligned} f_b &= f_1 + f_2 && \text{(string forces balanced by bridge force)} \\ v_b &= v_1 = v_2 && \text{(common velocity at bridge).} \end{aligned}$$

- The bridge impedance relates the force and velocity at the bridge as $F_b(\omega) = R_b(\omega)V_b(\omega)$.
- Expanding the traveling wave components in the frequency domain,

$$\begin{aligned} R_b(\omega)V_b(\omega) &= F_b(\omega) = F_1(\omega) + F_2(\omega) \\ &= [F_1^+(\omega) + F_1^-(\omega)] + [F_2^+(\omega) + F_2^-(\omega)] \\ &= R_1 \{V_1^+(\omega) - [V_b(\omega) - V_1^+(\omega)]\} \\ &+ R_2 \{V_2^+(\omega) - [V_b(\omega) - V_2^+(\omega)]\} \end{aligned}$$

or

$$V_b(\omega) = H_b(\omega) [R_1 V_1^+(\omega) + R_2 V_2^+(\omega)]$$

where R_i is the impedance of string i and

$$H_b(\omega) \triangleq \frac{2}{R_b(\omega) + R_1 + R_2}.$$

- In the time-domain, we scale each incoming string velocity, sum them together, and filter according to the transfer function $H_b(\omega) = 2/[R_b(\omega) + R_1 + R_2]$ to obtain the velocity of the bridge.
- The subsequent outgoing velocities are then found by

$$\begin{aligned} v_1^-(t) &= v_b(t) - v_1^+(t) \\ v_2^-(t) &= v_b(t) - v_2^+(t). \end{aligned}$$

- This analysis can be generalized to the case of N coupled strings without much difficulty.

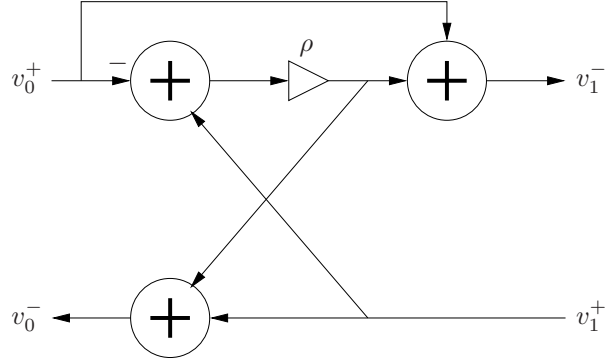


Figure 7: A one-multiply traveling-wave scattering junction block diagram.

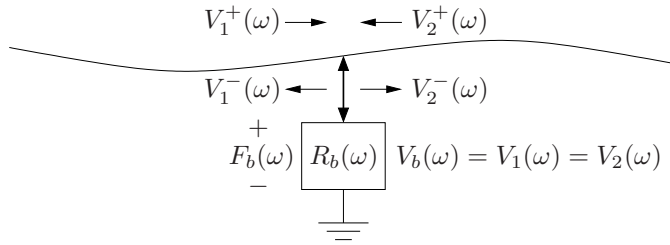


Figure 8: Two strings terminated at a common bridge impedance.

2.10 Coupled Dimensions of Vibration

- In real string instruments, wave motion is possible in at least four dimensions corresponding to horizontal and vertical transverse motion, longitudinal motion, and especially for bowed strings, torsional waves.
- If the string terminations were perfectly rigid, the horizontal and vertical transverse polarizations would be largely independent and we could model them with two identical, uncoupled, filtered delay loops, as shown in Fig. 9..
- For terminations that are yielding to some extent, there will be coupling between various planes of vibration. We can simulate linear coupling with digital filters that connect the two planes of vibration together (as shown in Fig. 10 below).
- If the coupling is symmetric, $H_{vh}(z) = H_{hv}(z)$.

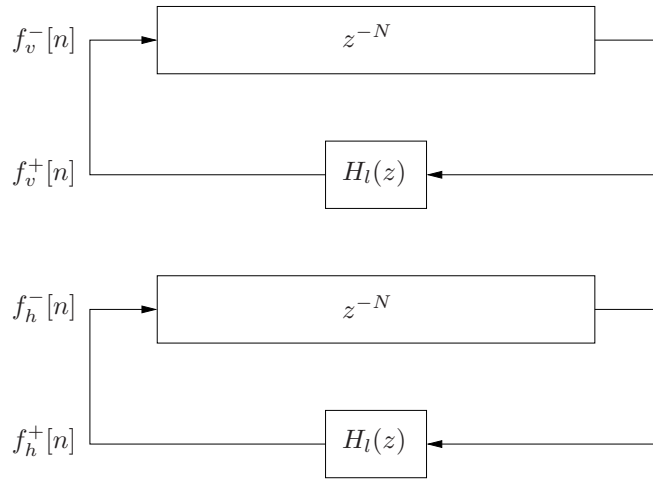


Figure 9: A digital waveguide simulation of a rigidly terminated string vibrating in two uncoupled planes of vibration.

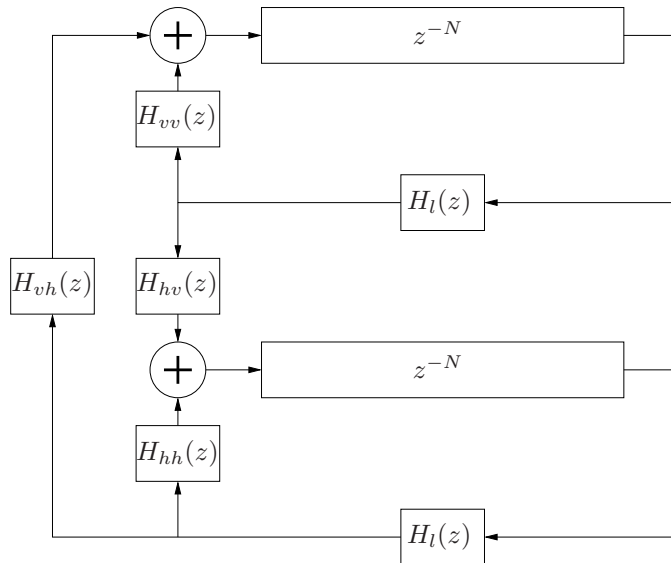


Figure 10: A digital waveguide string simulation with coupled horizontal and vertical planes of vibration.