MUMT 618: Week #7

1 Piano Modeling

In this section we consider further issues with respect to string instrument modeling in general, and the piano in particular. An excellent set of five lectures on the acoustics of the piano is available online.

1.1 Acoustic Background

1. **Aspects of Construction:**

   - Most pianos have 88 keys.

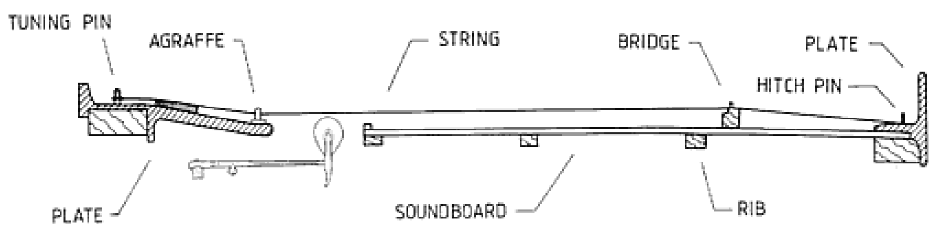

Figure 1: A simplified diagram of the piano (from "Five Lectures on the Acoustics of the Piano", edited by Anders Askenfelt).

   - A concert grand piano has 8 single wrapped strings, 5 pairs of wrapped strings, 7 sets of three wrapped strings, and 68 sets of three unwrapped strings.
   - To achieve a greater loudness, the piano strings are held at very high tensions. To offset these high forces, a sturdy cast iron frame is used.
   - The high string tensions demand high-strength wires.
   - Higher-strength wires are generally stiffer (or have a higher Young's modulus). This bending stiffness provides an additional restoring force (besides tension) that slightly raises the frequency of all the modes.
   - String bending is greater for the higher modes, resulting in greater frequency stretching at higher frequencies..
   - The resulting inharmonicity of strings is approximately given by

     $$f_n = nf_0\sqrt{1 + Bn^2},$$

     where f_n is the frequency of the nth harmonic and f_0 is the frequency of the fundamental. For a solid wire without wrapping,

     $$B = \frac{\pi^3 r^4 Y}{16TL^2},$$

     where r is the radius of the string, Y is the Young's modulus, T is the tension, and L is the length of the string.

- To minimize inharmonicities due to string stiffness, the smallest string diameter possible should be used (since B scales by r^4).
- In order to maintain string mass but minimize stiffness, the lower strings are wrapped.
- Because of the inherent inharmonicity of its strings, the piano is "stretch-tuned".

2. **Coupled Strings:**

- Over most of its playing range, the piano has three strings per note.
- In order to maximize decay time, these strings are slightly mistuned by about one to two cents from each other.
- The initial "in-phase" excitation of all three strings produces a rapid initial decay of string energy into the sound board. Because of mistunings, however, the vibrations soon grow out of phase and result in a much longer secondary decay.

3. **Hammer-String Interaction:**

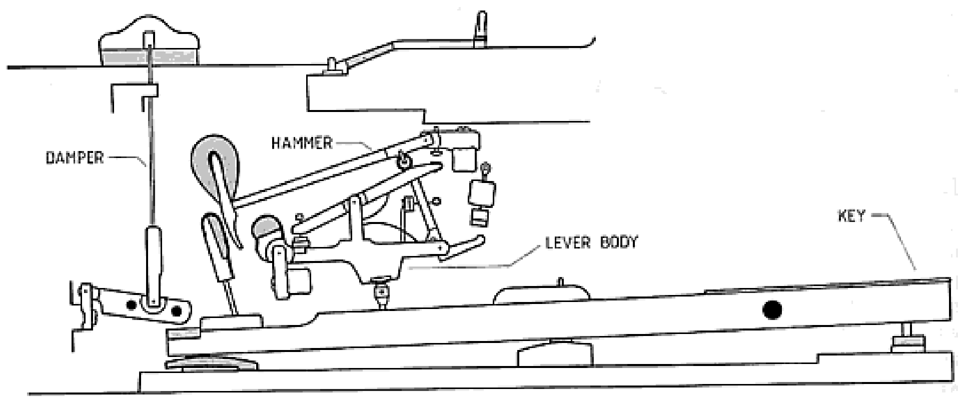

Figure 2: Modern grand piano action (from "From touch to string vibration" by Anders Askenfelt & Erik Jansson).

- The hammer action of the piano produces a "striking" excitation.
- The hammer is typically "thrown" away from the string by the first reflected pulse on the string. Depending on its weight, however, the hammer may remain in contact with the string for longer or shorter periods of time.

4. **The Soundboard:**

- The soundboard is nearly always made of spruce (of approximately 1 cm thickness) and is the main source of radiated sound.
- Strips of spruce are glued together and then ribs are added at right angles to the grain so that cross-grain stiffness is roughly equal to the natural stiffness along the grain.
- Modern pianos have two bridges.
- The lowest mode of a soundboard is typically around 50 Hz.

1.2 Dispersion in Strings

- The lossless one-dimension wave equation was previously derived for a string by assuming the restoring force due to string stiffness was negligible. However, it turns out that string stiffness cannot be ignored in many musical contexts.

- In general, strings of greater diameter have larger restoring forces due to bending.
- This restoring force can be accounted for in the wave equation with a term proportional to the fourth spatial derivative of the string displacement:

$$\epsilon \ddot{y} = T y'' - \kappa y'''',$$

where T is tension, $\kappa = Y \pi r^4/4$ is the moment constant for a cylindrical string of radius r and Young's modulus Y.

- This equation can be analyzed (see The Dispersive 1D Wave Equation for details) in order to estimate the resulting *frequency-dependent* wave velocity as:

$$c(\omega) = c_0 \left(1 + \frac{\kappa \omega^2}{2 T c_0^2} \right),$$

where $c_0 = \sqrt{T/\epsilon}$ is the lossless wave velocity.

- Higher frequency wave components travel with faster velocities, with the familiar result that the normal modes of the string are no longer perfectly harmonic.
- Because dispersion is particularly obvious in piano strings, a *stretched tuning* system is used in an attempt to minimize beating between simultaneous notes.

1.3 Dispersive Waveguide Modeling

- In a digital waveguide model, the effects of dispersion can be accounted for by considering the relationship between temporal and spatial sampling intervals:

$$X = c(\omega) T_s \qquad \Longrightarrow \qquad T_s(\omega) = \frac{X}{c(\omega)} = \frac{c_0 T_{s0}}{c(\omega)},$$

where T_{s0} is the unit delay time without dispersion.

- As a result, unit delays (z^{-1}) must be replaced by

$$z^{-1} \longrightarrow z^{-c_0/c(\omega)}.$$

- We can interpret this new delay unit as an *allpass filter* that approximates the corresponding frequency-dependent delay.
- In a digital waveguide model, dispersive wave propagation can thus be simulated by replacing unit delays with allpass filters, as illustrated in Fig. 3.

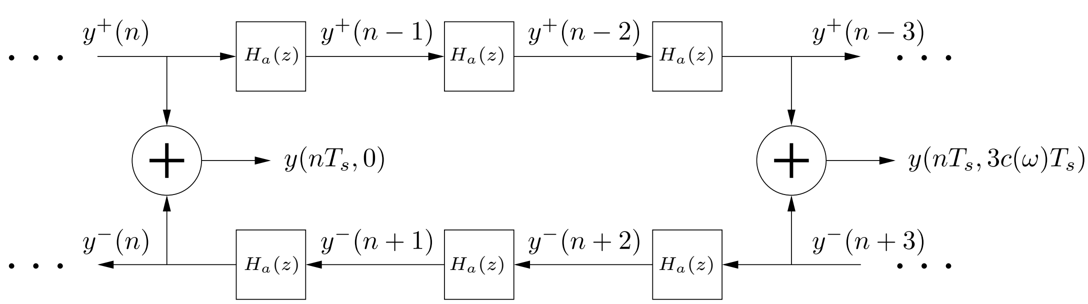

Figure 3: Discrete-time simulation of dispersive wave propagation, where digital allpass filters $H_a(z)$ replace unit delay elements.

- Because allpass filters are linear and time invariant, they can be commuted and implemented at discrete spatial locations in the model, in exactly the same way as previously discussed for other linear, time invariant gain factors or filters.
- In general, good approximations to dispersion require fairly high-order allpass filters. Contrary to the case with commuted frequency-dependent loss filters, the consolidation of allpass units does not necessarily lead to computational savings by way of lower-order filters.
- A number of approaches have been reported for dispersion filter design.
- To model stiff strings, the allpass response must exhibit a *decreasing* phase delay with increasing frequency.

1.4 High Note Modeling

- In general, digital waveguide models of string instruments make use of allpass delay-line interpolation techniques because FIR filter techniques result in too much high-frequency attenuation.
- The delay-line lengths necessary to produce very high fundamental frequencies may sometimes become too short for a given implementation. This problem could especially occur in conjunction with vectorized computational structures.
- The fundamental frequency of the highest piano note (C8) is 4186 Hz. At a sample rate of 44100 Hz, only four (unstretched) partials will possibly fall within the range of human hearing.
- Thus, for such high notes, one can instead model each partial with a separate second-order digital resonance filter. The model then consists of several resonance filters combined in parallel.

1.5 Hammer Input

- A simple one-dimensional digital waveguide string simulation is depicted in Fig. 4 below.

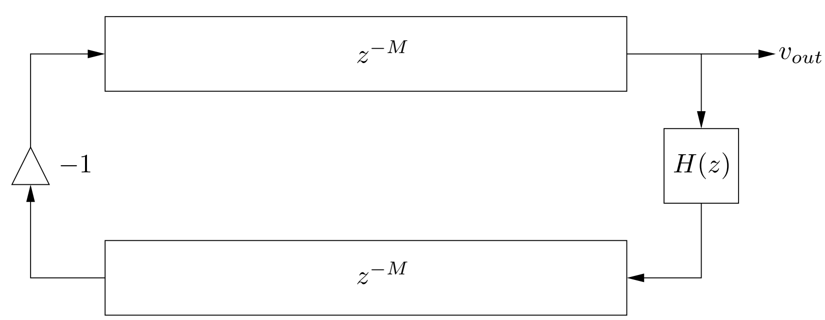

Figure 4: Discrete-time simulation of one-dimensional string vibrations.

- In previous sections, we modeled displacement or velocity excitations by appropriate delay-line initializations.
- Piano strings are excited by hammer strikes, which produce velocity input profiles. When the hammer hits (and remains in contact with) the string, the string is effectively divided into two parts.
- Based on this simple analysis, a more physical implementation for a piano synthesis system is diagrammed in Fig. 5, where the delay-line length $M = M_1 + M_2$.
- It is possible to rearrange the block diagram components of Fig. 5 to produce the equivalent system shown in Fig. 6.

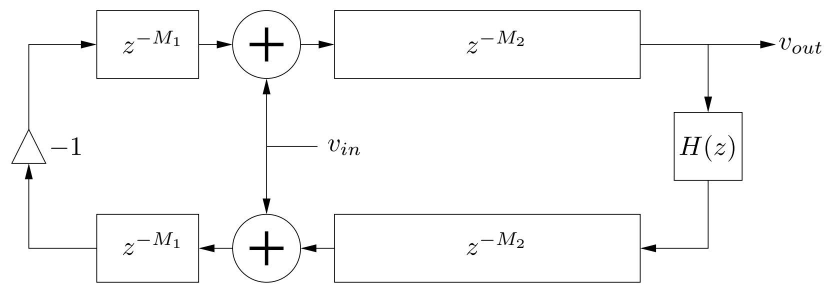

Figure 5: Hammer input in a discrete-time simulation of a piano string.

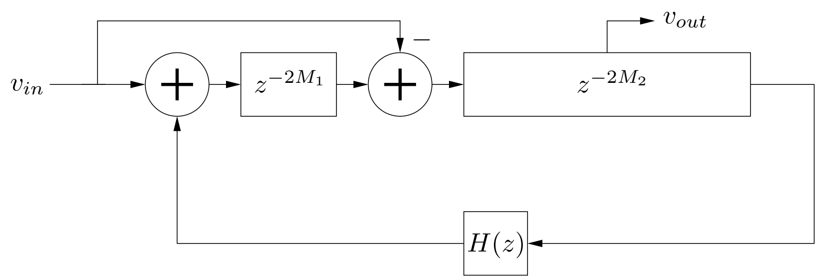

Figure 6: Equivalent system depicting hammer input in a discrete-time simulation of a piano string.

- These block-diagram components can be further re-distributed to produce the equivalent system shown in Fig. 7.
- Figure 7 makes explicit the fact that a feedforward comb filter can be placed in series with a digital waveguide string simulation to account for the effects of pluck position.

1.6 A Commuted Hammer Model

- When the piano hammer strikes the string(s), a pulse travels to the agraffe and back. As long as the hammer remains in contact with the string, there will be subsequent pulses and reflections.
- The piano hammer is typically "thrown away" from the string by one of the reflected pulses.
- A plot depicting a theoretical hammer-string interaction force is shown in Fig. 8. This plot corresponds to three pulse reflections.
- The actual number of pulse reflections that occur is dependent on the string being hit and the initial velocity of the hammer.
- The hammer-string interaction is inherently non-linear, in large part because of the compression characteristics of the hammer felt.
- However, because the pulse-hammer interactions occur at discrete times, it is possible to consider them as distinct events that overlap to produce a complete hammer-string characteristic.
- Each single pulse can be modeled as a lowpass filtered impulse signal. This suggests a hammer-excited piano synthesis system as diagrammed in Fig. 9.

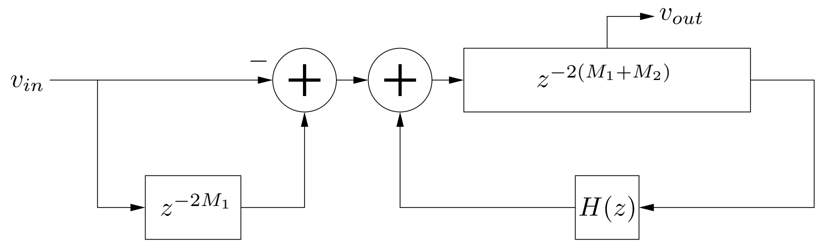

Figure 7: Another equivalent piano string simulation with comb-filtered input factored out of the string feedback loop.

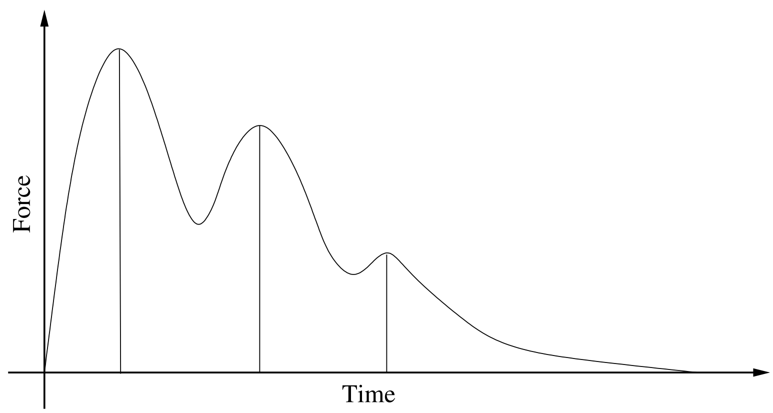

Figure 8: A theoretical hammer force characteristic.

- In the system of Fig. 9, an input velocity control triggers an impulse signal. The tapped delay-line is used to produce three subsequent time-delayed impulses, each of which drive a particular lowpass filter. The resultant linear summation of time-delayed lowpass filter impulse responses then drives the string model and its soundboard response filter.

- Hammer-string characteristics can be evaluated for various strings and input velocities and appropriate lowpass filter responses designed. The input velocity can then be used as a control parameter to select the appropriate filter characteristics.

- Finally, because the implementation described by Fig. 9 is linear, the piano soundboard impulse response can be "commuted" through the system and used to trigger the hammer-string filters as shown in Fig. 10.

2 The Wave Digital Piano Hammer

In this section, we use a model of the piano hammer to introduce wave digital filtering techniques. This approach to modeling the piano hammer was developed by Scott Van Duyne, John Pierce, and Julius Smith and reported in "Traveling wave implementation of a lossless mode-coupling filter and the wave digital hammer," *Proceedings of the International Computer Music Conference*, Århus, Denmark, 1994.

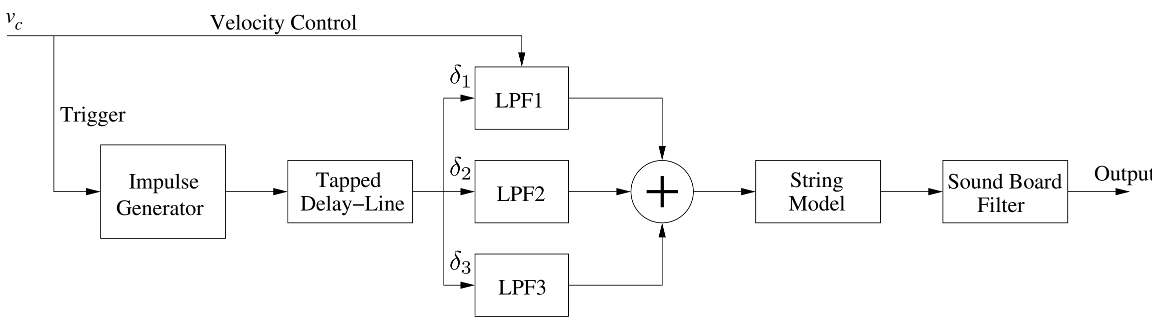

Figure 9: A hammer-excited piano synthesis block diagram implementation.

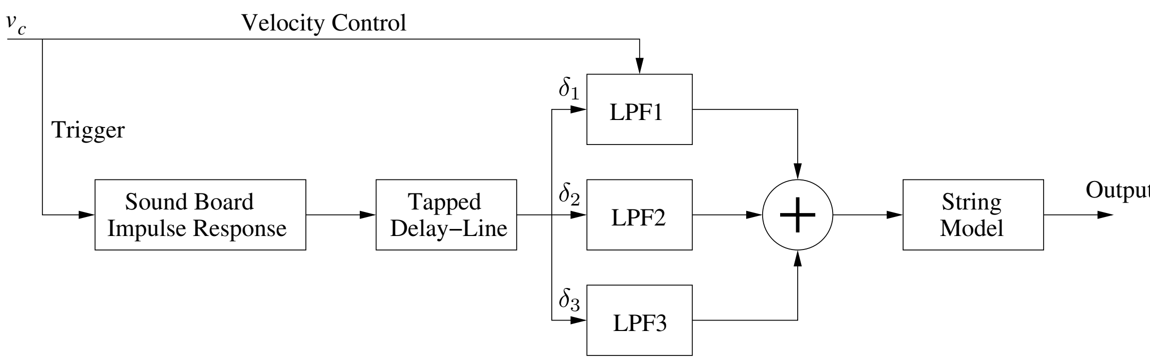

Figure 10: A commuted piano hammer block-diagram implementation.

2.1 Wave Digital Filter Background

- Wave digital filters (WDF) were initially developed by Alfred Fettweis in the late 1960s for digitizing lumped electrical circuits.
- Wave digital filter techniques involve the use of traveling-wave variables in the context of lumped system analysis.
- Very simple expressions can be developed for various physical components, such as masses, springs, and dashpots. These components can then be interconnected using adaptors, which are scattering junction interfaces.
- WDF techniques are convenient when trying to eliminate delay-free loops because of the flexibility available in defining system wave impedances.

  A WDF model of a system can be developed from its continuous-time, differential equation representation as follows:

  1. Express all physical quantities, such as forces and velocities, in terms of traveling-wave components.
  2. Digitize the resulting traveling-wave system using the bilinear transform.
  3. Combine elementary components (mass, springs, ...) using scattering junctions defined by either series or parallel connections.

- For a more detailed and complete discussion, see the following wave digital filter link.

2.2 The Hammer Model

- The piano hammer, or a felt mallet, can be modeled as a non-linear spring and mass system.

- The spring represents the felt portion of the hammer or mallet and its compliance is considered to vary with felt compression. In this way, the felt is modeled as a non-linear spring with a stiffness "constant" that increases as it is compressed.

- First consider a linear mass and spring system as shown in Fig. 11. The mass is assumed to have an initial velocity at time zero.

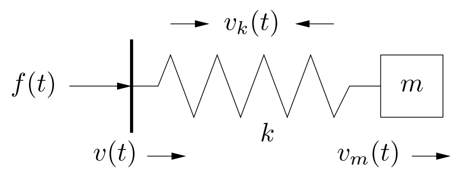

Figure 11: A mass and spring force diagram.

- Since the mass and spring are in parallel, the driving force applied on the left side of the spring, f, is equally applied to the spring and to the mass:

$$f = f_k = f_m.$$

- The driving point velocity, v, is equal to the sum of the spring compression velocity and the mass velocity,

$$v = v_k + v_m.$$

- By Hooke's law, the restoring force of the spring is given by

$$f_k(t) = kx(t) \qquad \Longrightarrow \qquad \frac{df_k(t)}{dt} = kv_k(t),$$

  where $x(t)$ is the compression distance of the spring and k is the spring stiffness constant.

- By Newton's law, the force of the mass is

$$f_m(t) = m\frac{dv_m(t)}{dt} - mv_0\delta(t),$$

  where $mv_0\delta(t)$ represents an initial momentum impulse setting the mass in motion with velocity v_0.

- Both of these force equations can be expressed in terms of their Laplace transforms and written as a driving-point admittance:

$$\begin{aligned} V(s) &= V_k(s) + V_m(s) \\ &= \frac{s}{k}F_k(s) + \frac{1}{ms}F_m(s) + \frac{v_0}{s}, \\ &= \left(\frac{s^2 + k/m}{ks}\right) F(s) + \frac{v_0}{s}. \end{aligned}$$

  The admittance function, $(s^2 + k/m)/ks$, is the steady-state response of the system with zeros at $\pm\sqrt{k/m}$. The v_0/s term represents the transient effect of the momentum impulse applied at time zero.

2.3 Traveling-Wave Decomposition of the Lumped System

- The lumped driving-point admittance of the mass-spring system can be defined in terms of wave variables in a manner analogous to distributed systems.

- An applied force can be written as $f = f_{\text{in}} + f_{\text{out}}$, where f_{in} is viewed as a force wave traveling into the admittance and f_{out} is viewed as a force wave traveling out of the admittance.
- In a similar way, we can define $v = v_{\text{in}} + v_{\text{out}}$.
- In distributed systems, waves have a finite propagation speed and force and velocity wave components can be related by a wave impedance, which is defined by the parameters of the medium of travel.
- In lumped systems, however, traveling waves may only be understood to travel instantaneously and there is no particular medium. But an arbitrary reference impedance, R_h, can still be defined such that:

$$F_{\text{in}} = R_h V_{\text{in}} \qquad F_{\text{out}} = -R_h V_{\text{out}}.$$

- Making this change of variables, the velocity transfer function can be written

$$V_{\text{out}}(s) = \left(\frac{s^2 - (k/R_h)s + k/m}{s^2 + (k/R_h)s + k/m} \right) V_{\text{in}}(s) + \frac{v_0 k/R_h}{s^2 + (k/R_h)s + k/m}.$$

- Using the bilinear transform to convert from a continuous-time system to a discrete-time system, we obtain

$$V_{\text{out}}(z) = \left(\frac{b_0 + b_1 z^{-1} + z^{-2}}{1 + b_1 z^{-1} + b_0 z^{-2}} \right) V_{\text{in}}(z) + \frac{\frac{v_0 mk}{T_s(mR_h\alpha^2 + mk\alpha + kR_h)}(1 + 2z^{-1} + z^{-2})}{1 + b_1 z^{-1} + b_0 z^{-2}},$$

where the continuous-time impulse function mapping has been bandlimited as $\delta(t) \to \delta(n)/T_s$ and the coefficients are defined as

$$\begin{aligned} b_0 &\triangleq \frac{mR_h\alpha^2 - mk\alpha + kR_h}{mR_h\alpha^2 + mk\alpha + kR_h}, \\ b_1 &\triangleq \frac{2(kR_h - mR_h\alpha^2)}{mR_h\alpha^2 + mk\alpha + kR_h}. \end{aligned}$$

- Because the reference impedance R_h is arbitrary, it is possible to choose a value such that the coefficient $b_0 = 0$, as

$$R_h = \frac{mk\alpha}{m\alpha^2 + k}.$$

- In this way, a unit of delay can be factored out of the second-order transfer function, such that

$$V_{\text{out}}(z) = z^{-1} H(z) V_{\text{in}}(z) + G(z) v_0,$$

where

$$\begin{aligned} H(z) &= \frac{a_0 + z^{-1}}{1 + a_0 z^{-1}}, \\ G(z) &= \left(\frac{1}{2\alpha T_s} \right) \frac{1 + 2z^{-1} + z^{-2}}{1 + a_0 z^{-1}}, \end{aligned}$$

with a_0 given by

$$a_0 = \frac{k - m\alpha^2}{k + m\alpha^2}.$$

- It is then possible to rewrite R_h in terms of a_0, eliminating k, as

$$R_h = \frac{m\alpha}{2}(1 + a_0).$$

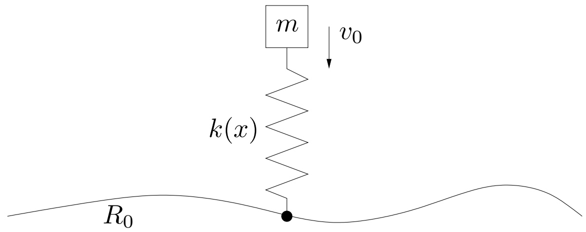

Figure 12: A string loaded by a mass and spring.

2.4 The Scattering Junction Connection

- A piano hammer strikes the string and then remains in contact with it for a certain time duration (before being thrown off by forces propagating in the string).
- During the time the hammer is in contact with the string, the string is "loaded" by the hammer impedance and wave scattering occurs.
- A lossless three-port scattering junction, as shown in Fig. 13, can be developed that defines the energy transfer between the hammer and the two sections of string.

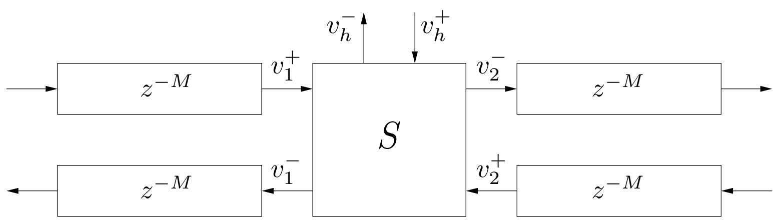

Figure 13: A three-port hammer/string scattering junction.

- At the junction, the velocities of the string sections and the hammer spring must all be equal,

$$v_1 = v_2 = v_h \stackrel{\Delta}{=} v_J.$$

- In addition, the sum of the forces at the junction must be zero, since it is a massless point,

$$f_1 + f_2 + f_h \stackrel{\Delta}{=} f_J = 0.$$

- Using these expressions and the previous wave variable definitions, the scattering equations become:

$$\begin{aligned} v_J &= \frac{2(R_0 v_1^+ + R_0 v_2^+ + R_h v_h^+)}{2R_0 + R_h} \\ v_1^- &= v_J - v_1^+ \\ v_2^- &= v_J - v_2^+ \\ v_h^- &= v_J - v_h^+ \end{aligned}$$

2.5 Making the Felt Nonlinear

- As originally discussed, the felt is modeled as a nonlinear spring with a spring "constant" that increases as the compression of the spring increases. In this way, the spring stiffness constant $k(x_k)$ is dependent on x_k.

- Because $x_k = f_k/k$ and because the driving-point force on the mass and spring system is equal to f_k, we can write

$$\begin{aligned} x_k &= (1/k)(f_h^- + f_h^+) \\ &= (R_h/k)(v_h^- - v_h^+). \end{aligned}$$

- The above expression can then be written in terms of a_0 as

$$x_k = \frac{1 - a_0}{2\alpha}(v_h^- - v_h^+).$$

- At any time in the model, the value of x_k can be used to lookup or determine an appropriate value for the spring constant k.

2.6 The Lossless Digital Model

- The resulting system of equations can be combined as shown in the block diagram of Fig. 14.

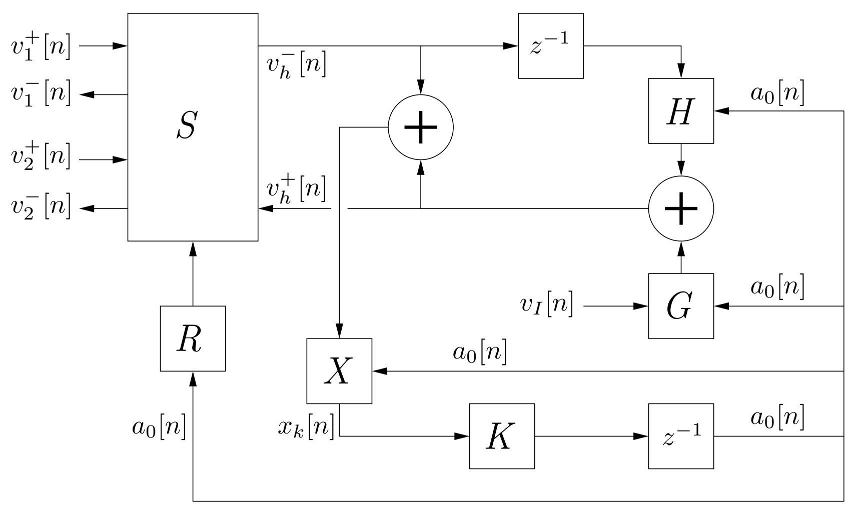

Figure 14: The wave digital hammer block diagram.

- S is the three-port scattering junction.
- H is a time-varying allpass filter.
- G is the modified integrator that integrates the incoming initial hammer velocity impulse signal, $v_I[n]$, taking into account the wave decomposition and the nonlinear effects of the hammer felt.
- X computes the actual felt compression, taking advantage of the fact that the force of compression on the felt is instantaneously proportional to the felt compression.
- K is a look-up table for the stiffness coefficient, $a_0[n]$, indexed by felt compression.
- R computes the time-varying wave impedance of the hammer system from a_0.
- $R_h[n] = 0$ when the hammer is not touching the string and $k(x_k) = 0$ as well, so the string equations are then not affected by the hammer system.