MUMT 618: Week #5

1 String Instrument Acoustics

As we will see in this and subsequent sections, most musical instruments can be analyzed in terms of three principle vibrational components: a waveguide, an output filtering system, and an energy input mechanism. The corresponding elements of a general string instrument are its strings, body, and plucking, striking, or bowing techniques, respectively.

1.1 The String as a Waveguide

- The uniform stretched string forms a one-dimensional waveguide along which mechanical waves can travel with a constant speed of propagation (ignoring for the moment other dimensions of wave propagation).
- The time evolution of the string's shape is given by the linear superposition of traveling-wave components propagating in opposite directions along the string (Matlab example).
- The resonance frequencies (or normal modes) of the string are determined by its length, the wave speed of propagation, and the boundary conditions at each of its ends.
- For a string of length L, fixed at each of its ends, a discrete set of standing-wave patterns is possible with frequencies given by integer multiples of $f_0 = c/(2L)$, where c is the speed of wave propagation on the string. These standing-wave frequencies correspond to the resonances of the system.
- The extent to which the resonance frequencies of the string are present in any particular string vibration pattern is determined by the way the system is driven or excited.
- The stiffness of the string produces a restoring force that was neglected in our derivation of the wave equation. When we account for this force, the string modes are "stretched" from perfect harmonic relationships (with greater stretching for higher modes).
- For thin metal strings, the decay time is determined mostly by air viscosity. For gut or nylon strings, internal damping is dominant for most modes.

1.2 String Energy Input Mechanisms

1. **The Plucked String:**
   - Plucking a string provides it with an initial energy displacement (potential energy).
   - The shape of the string before its release completely defines the harmonic signature of the resulting motion.
   - A string plucked at $1/n$th the distance from one end will not have energy at the nth partial and its integer multiples.
   - The strength of excitation of the nth vibrational mode is inversely proportional to the square of the mode number.

2. **The Struck String:**

   - A struck string is given an initial velocity distribution (kinetic energy).
   - A string struck at $1/n$th the distance from one end will not have energy at the nth partial and its integer multiples.
   - The harmonic amplitudes in the vibration spectrum of a struck string fall off less rapidly with frequency than those of plucked strings.
   - Light "hammers" (mass much less than the mass of the string) result in little spectral drop-off with frequency. Heavier hammers produce a drop-off roughly proportional to the inverse of the mode number.

3. **The Bowed String:**

   - "Stick-slip" mechanism: During the greater part of each period, the string is "stuck" to the bow and is carried with it in its motion. Then the string suddenly detaches itself and moves rapidly backward until it is caught again by the moving bow.
   - Beginning and end of the slipping are triggered by the arrival of the propagating bend or "kink".
   - The string's vertical motion at any one point is given by a sawtooth pattern.
   - Round trip time depends only on the string length and the wave velocity.
   - Bowing near the string end requires greater force and produces a louder, brighter sound than bowing farther from the end.
   - Amplitude of vibration can be increased either by increasing the bow speed or by bowing closer to the bridge.

1.3 String Instrument Bodies

1.3.1 The Violin Body

The violin is probably the most studied musical instrument today, with important analyses dating back over 100 years and including contributions from Helmholtz and the Nobel laureate C.V. Raman [Helmholtz, 1954, Raman, 1918].
An excellent review of the acoustics of the violin is provided by Woodhouse [2014]. Another excellent review is provided by Gough [2016].

1. **Air and Wood Modes:**

   - For a violin, there are typically three or four important body resonances below 1 kHz. These include the first air mode, the *T1* top plate mode, and the third and fourth "corpus" modes.
   - Above 1 kHz, the mode structure is usually difficult to decipher, though a concentration of resonances around 2 to 3 kHz (the "bridge hill") appears to be important (related to perception).

2. **Tuning Top and Back Plates:**

   - Various techniques exist to measure the modal frequencies of the various components (Chladni patterns, force hammers, hologram interferometry).
   - Complete systems, however, will have different mode structures than the individual plates.
   - Colin Gough has contributed a number of research papers that help explain the transition from free plate modes to complete violin-like body modes [Gough, 2015b,a, 2000].

3. **The Bridge:**

   - The bridge transforms the motion of the vibrating string into a driving force on the top plate of the instrument.
   - The violin bridge typically has strong resonances around 3000 and 6000 Hz.

- The bridge must move a small amount in order to transfer energy from the string to the body. For most musical instruments, however, the rate of energy transfer from the string to the bridge and soundboard is quite small (energy decay in the string is most affected by air viscosity and internal string damping).

1.3.2 Other Stringed Instruments

1. **Guitars:**
   - At low frequencies, sound radiation occurs from both plates. At high frequencies, however, most of the radiated sound comes from the top plate.
2. **Electric Guitars:**
   - The body has less effect on the sound. Electrical circuits replace the body filtering function.

2 String Modeling

In this section, we investigate digital waveguide methods for modeling simple string instruments.

2.1 Ideal Rigid and Free Terminations

- At a fixed end, the string's transverse displacement is zero. Consider the general traveling wave solution to the wave equation: $y(t,x) = y^+(ct-x) + y^-(ct+x)$. If the string is fixed at $x = 0$, then $y(t,0) = 0$ and $y^+(ct) = -y^-(ct)$, which indicates that displacement traveling waves reflect from a fixed end with an inversion (or a *reflection coefficient* of -1).
- Since the displacement at a rigid termination is always zero, the physical velocity must also be zero for all time. Therefore, traveling-wave components of velocity will also reflect with a reflection coefficient of -1 at such a boundary.
- Force and velocity traveling-wave components are related by the wave impedance as $f^+ = Rv^+, f^- = -Rv^-$. At a rigid terminiation, $v^+ = -v^-$ (from above). Thus, force wave components can be related at a rigid termination as:
  $$f^+ = Rv^+ = R(-v^-) = -Rv^- = f^-.$$
  From this, we see that traveling-wave components of force are related by a reflection coefficient of +1 at a rigid termination.
- At a free end, $\partial y/\partial x = 0$ because no transverse force is possible. At such a boundary, traveling-wave components of force must reflect with a coefficient of -1. To determine the reflection coefficient for displacement waves, we first note that force waves are proportional to the string slope. Differentiation of our general traveling-wave solution by x leads to (see this link):
  $$\frac{\partial}{\partial x}y(t,x) = 0 = -\frac{1}{c}\frac{\partial}{\partial t}y^+(t-x/c) + \frac{1}{c}\frac{\partial}{\partial t}y^-(t+x/c),$$
  After integrating this expression with respect to time, we have $y^+(t) = y^-(t)$ at $x = 0$, indicating that displacement traveling waves reflect with a coefficient of +1.

2.2 The Ideal Plucked String

- The simulation of displacement wave motion in a string rigidly terminated at both its ends (and without losses) is shown in the figure below.
- An ideal plucked string is defined as having an initial displacement and zero initial velocity. In the model, the delay lines should be initialized with displacement data corresponding to some arbitrary initial string shape (as illustrated in the figure above by the dashed lines).

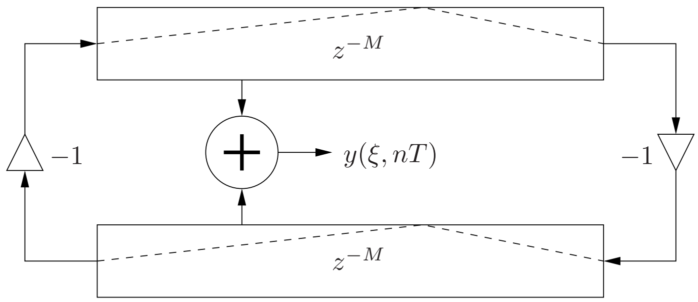


Figure 1: Digital waveguide simulation of ideal lossless wave propagation on a string fixed at both ends (with pluck initialization).

- Because the physical displacement of a string is given by the superposition of left- and right-going traveling waves, the initial amplitude of each delay-line section should be half the amplitude of the initial, physical string displacement.
- The initial displacement shape must be bandlimited to half the discrete-time sample rate. Because sharp corners imply an infinite bandwidth, plucking points should be rounded to some extent. In fact, this is physical given the stiffness of real strings and the finite size of plectra.
- It is possible to use digital waveguide models to simulate other wave variables as well, such as velocity or acceleration waves. Note that an ideal pluck shape corresponds to positive and negative acceleration impulses.
- The simulation of velocity wave motion in a string rigidly terminated at both its ends (and without losses) is shown in the figure below (for the pluck displacement shown above).

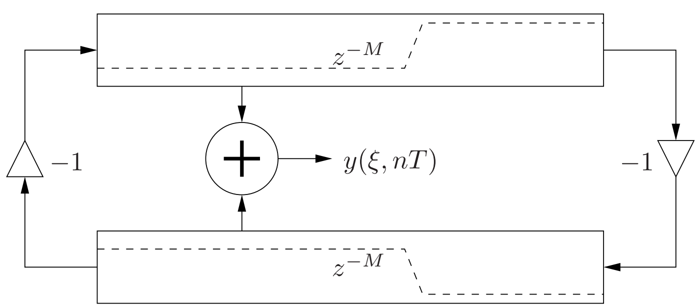


Figure 2: Digital waveguide velocity-wave simulation of ideal lossless wave propagation on a string fixed at both ends (with pluck initialization).

- For the moment, we ignore the fact that the string vibrations are influenced and filtered by the body resonances before being transmitted into the air.

2.3 The Ideal Struck String

- An ideal struck string involves zero initial displacement and a nonzero initial velocity distribution.
- The simulation of velocity wave motion in a struck string rigidly terminated at both its ends (and without losses) is shown in the figure below.

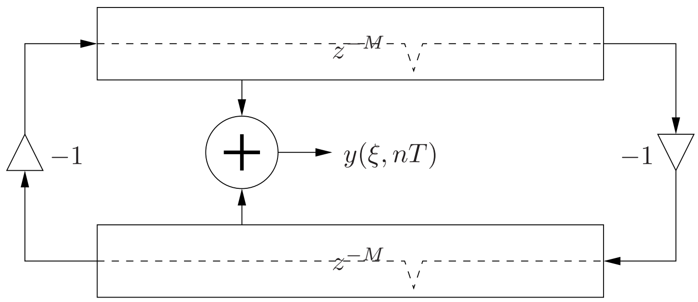


Figure 3: Digital waveguide velocity-wave simulation of ideal lossless wave propagation on a string fixed at both ends (with strike initialization).

- For the struck string, simulation of velocity waves is more natural. Alternately, the initial velocity distribution could be integrated with respect to x from $x = 0$, divided by c, and negated in the upper rail to obtain the equivalent initial displacement (since $v^{\pm} = \pm f^{\pm}/R = \mp c y'^{\pm}$, where R is the wave impedance of the string).
- The simulation of displacement wave motion in a struck string rigidly terminated at both its ends (and without losses) is shown in the figure below (for the strike initialization shown above).

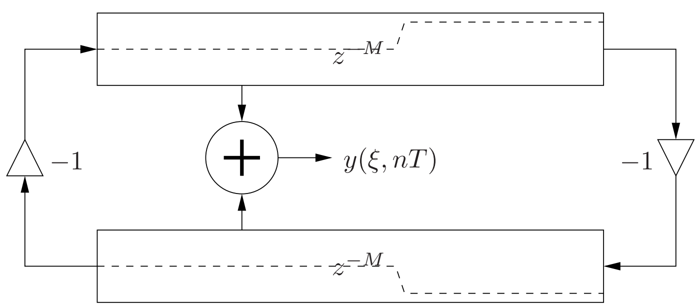


Figure 4: Digital waveguide displacement-wave simulation of ideal lossless wave propagation on a string fixed at both ends (with strike initialization).

2.4 Damping & Stiffness

- Real wave propagation along strings involves losses and dispersion.
- Propagation losses are generally frequency-dependent. In general, damping will increase with frequency.
- Losses in strings have been investigated by Valette [1995], with the main mechanisms involving the viscous effect of the surrounding air, viscoelasticity and thermoelasticity of the string material, and internal friction representing both macroscopic rubbing (in multi-stranded and overwound strings) and inter-molecular effects (in monofilament strings).
- Frequency-dependent losses could be determined on a per sample basis and implemented between each sample of delay in a digital waveguide model. However, such a scheme would be inefficient and the design of appropriate digital filters could be difficult.
- Instead, we take advantage of the linear and time-invariant nature of the waveguide structure to commute and implemented the losses at just a few locations within the model.

- In order to maintain stability (passivity), $\left|\hat{G}(e^{j\omega T})\right| \leq 1$.
- A string fixed at both ends can be implemented with two delay lines, a discrete-time filter $\hat{G}(z)$ representing losses and inversion at the bridge, and an inversion representing reflection from the nut, as shown below. Additional simplifications are possible.

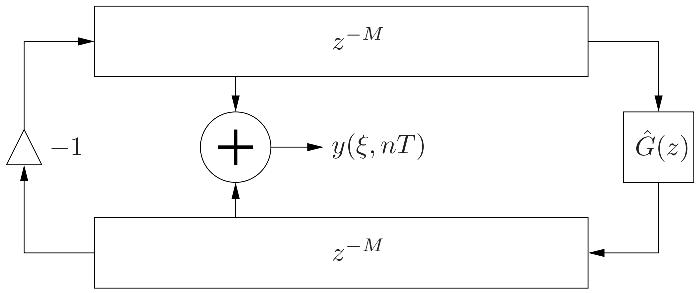


Figure 5: Digital waveguide simulation of dispersive wave propagation on a string fixed at both ends.

- String stiffness produces dispersion, or a variation of wave propagation speed with frequency. The use of an allpass filter in the string loop is effective in accurately simulating such dispersion.

2.5 Simulating Pluck/Strike Position

- From string acoustics, we know that a string plucked at $1/n$th the distance from one end will not have energy at the nth partial and its integer multiples.
- If we initialize the delay lines of a waveguide model with a particular plucked string shape, the pluck position "filtering" is automatically included (it's defined by the initial energy profile that is defined by the initial shape).
- In practice, however, it is common that the delay lines are either initialized with noise or driven with an external input signal (in which case they are initialized with zeros).
- Pluck position filtering can be simulated with a feedforward comb filter, which creates notches in the frequency magnitude response at equal frequency intervals.

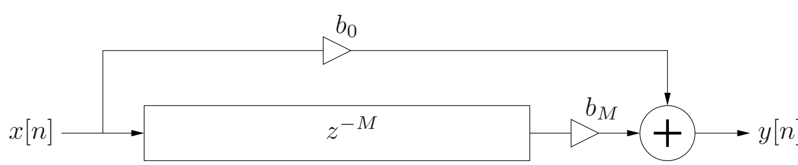


Figure 6: A feedforward comb filter block diagram.

- For $b_M > 0$, there are M peaks in the frequency response, centered about the frequencies $\omega_k = 2\pi k/M, k = 0, 1, \ldots, M-1$. Between these peaks, there are M notches at intervals of f_s/M Hz.

2.6 Commuted Waveguide Synthesis

Plucked string instruments are well modeled as linear systems and this allows great flexibility in the way they are implemented. Commuted synthesis takes advantage of this fact and provides a highly efficient way of producing high quality plucked-string sounds.

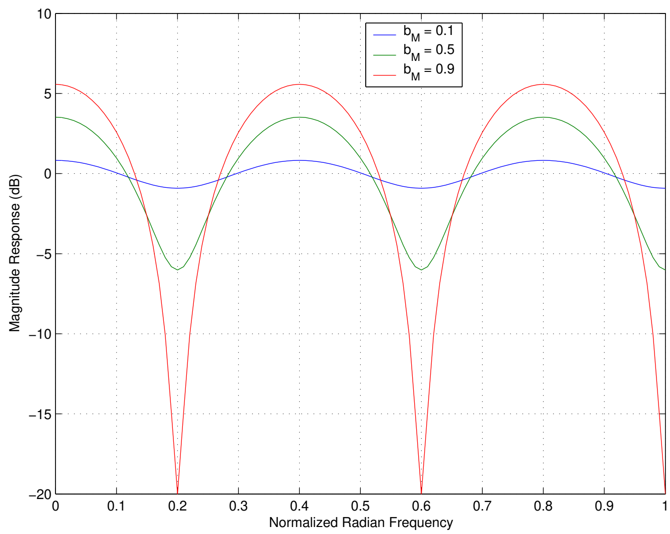
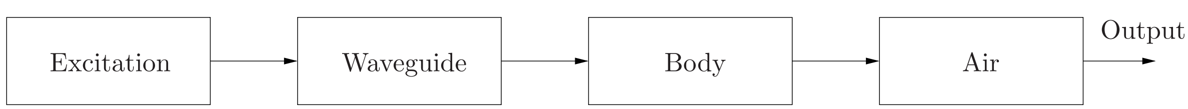


Figure 7: Magnitude response of a feedforward comb filter with $M = 5$, $b_0 = 1$, and $g = b_M = 0.1$, 0.5, and 0.9.

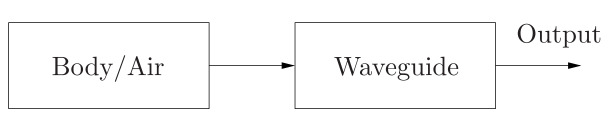
- The model components discussed thus far only account for vibrations of the string. In order to convincingly simulate a complete string instrument system, we must also account for the bridge and body response, as well as radiation patterns into the surrounding environment.
- Assuming we could develop representative filters for the various components of a string instrument, the output response of the complete system would be diagrammed and computed as shown below:

"Classic Model"



- For plucked or ideally struck strings, each component of the system is linear and time-invariant. Therefore, the system components can be rearranged without affecting the overall output.
- The body response of a string instrument is complex and generally requires a high-order digital filter to accurately simulate. However, by taking advantage of system commutativity, it is possible to record a body "impulse response" and use it as the string excitation. This technique is referred to as *commuted waveguide synthesis*. As shown, it is also possible to commute the radiation directivity response.

"Commuted Synthesis Model"



- When using commuted synthesis, body size can be roughly simulated with the body response playback rate, which shifts the body resonances higher or lower in frequency. This technique is demonstrated in the STK `Mandolin` class, which implements commuted synthesis of a two string mandolin instrument.

- For commuted synthesis, it is possible to pre-filter the input signal with a given pluck position filter. To maintain synthesis parameter flexibility, this is typically done during the synthesis computation.

2.7 Coupled Strings

- Figure 8 depicts the case of two strings that terminate at a common, non-rigid bridge of impedance $Z_b(\omega)$.

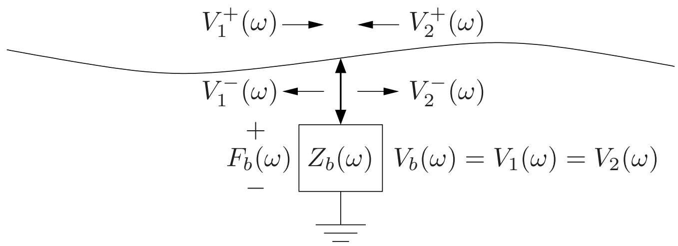


Figure 8: Two strings terminated at a common bridge impedance.

- The constraints at the junction are:

$$\begin{aligned} f_b &= f_1 + f_2 && \text{(string forces balanced by bridge force)} \\ v_b &= v_1 = v_2 && \text{(common velocity at bridge).} \end{aligned}$$

- The bridge impedance relates the force and velocity at the bridge as $F_b(\omega) = Z_b(\omega)V_b(\omega)$.
- Expanding the traveling wave components in the frequency domain,

$$\begin{aligned} Z_b(\omega)V_b(\omega) &= F_b(\omega) = F_1(\omega) + F_2(\omega) \\ &= \left[F_1^+(\omega) + F_1^-(\omega)\right] + \left[F_2^+(\omega) + F_2^-(\omega)\right] \\ &= R_1\left\{V_1^+(\omega) - \left[V_b(\omega) - V_1^+(\omega)\right]\right\} \\ &+ R_2\left\{V_2^+(\omega) - \left[V_b(\omega) - V_2^+(\omega)\right]\right\} \end{aligned}$$

or

$$V_b(\omega) = H_b(\omega)\left[R_1V_1^+(\omega) + R_2V_2^+(\omega)\right]$$

where R_i is the impedance of string i and

$$H_b(\omega) \triangleq \frac{2}{Z_b(\omega) + R_1 + R_2}.$$

- In the time-domain, we scale each incoming string velocity, sum them together, and filter according to the transfer function $H_b(\omega) = 2/\left[Z_b(\omega) + R_1 + R_2\right]$ to obtain the velocity of the bridge.
- The subsequent outgoing velocities are then found by

$$\begin{aligned} v_1^-(t) &= v_b(t) - v_1^+(t) \\ v_2^-(t) &= v_b(t) - v_2^+(t). \end{aligned}$$

- This analysis can be generalized to the case of N coupled strings without much difficulty.

2.8 Coupled Dimensions of Vibration

- In real string instruments, wave motion is possible in at least four dimensions corresponding to horizontal and vertical transverse motion, longitudinal motion, and especially for bowed strings, torsional waves.
- If the string terminations were perfectly rigid, the horizontal and vertical transverse polarizations would be largely independent and we could model them with two identical, uncoupled, filtered delay loops, as shown in Fig. 9.

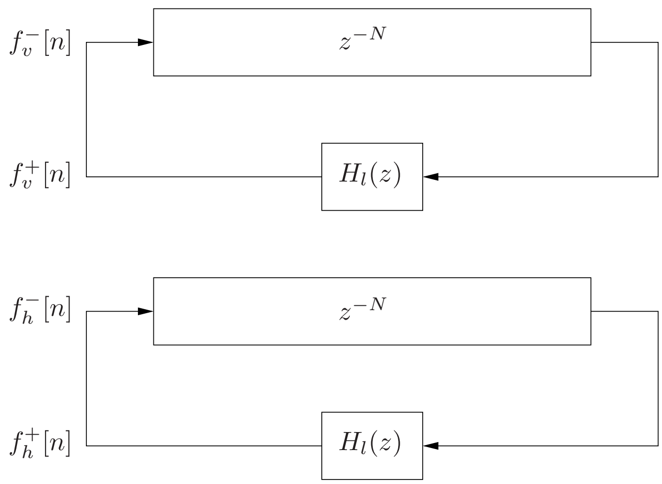


Figure 9: A digital waveguide simulation of a rigidly terminated string vibrating in two uncoupled planes of vibration.

- For terminations that are yielding to some extent, there will be coupling between various planes of vibration. We can simulate linear coupling with digital filters that connect the two planes of vibration together (as shown in Fig. 10 below).

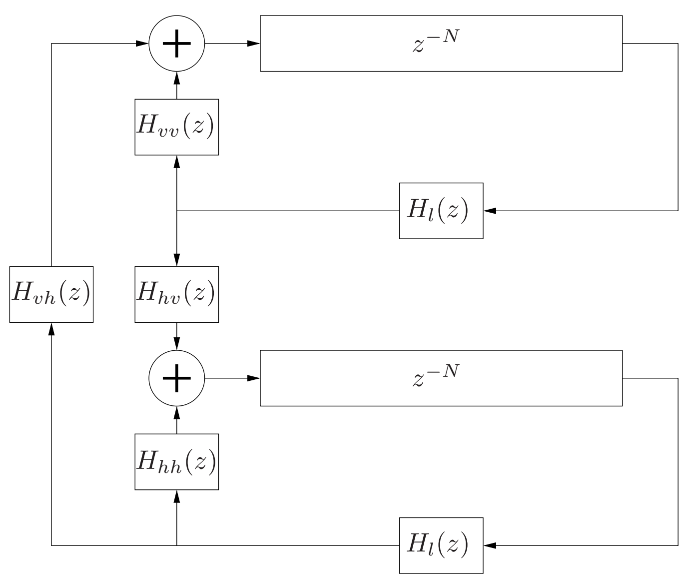


Figure 10: A digital waveguide string simulation with coupled horizontal and vertical planes of vibration.

- If the coupling is symmetric, $H_{vh}(z) = H_{hv}(z)$.

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