MUMT 618: Week #6

1 Lumped System Analysis:

Sound is caused by vibrating objects or media. In this section, we study somewhat overly simplified, ideal structures to gain an understanding of the fundamental concepts of vibrating systems. The analysis of systems in terms of finite collections of masses, springs, and loss mechanisms is referred to as lumped modeling and the resulting approximate system responses are called "lumped characterizations".

1.1 Lumped Elements:

1. **The Ideal Dashpot:**

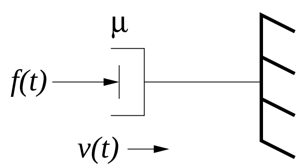

Figure 1: An ideal dashpot or damper.

- The force necessary to overcome a mechanical dashpot or resistance is typically approximated as being proportional to velocity: $f(t) = \mu v(t) = \mu \frac{dx}{dt}$
- The dashpot impedance is simply given by μ, the *frequency-independent* damping constant.
- The electrical correlate of a mechanical dashpot is a resistor, characterized by $v(t) = Ri(t)$. In an analog equivalent circuit, a dashpot can be represented by a resistor $R = \mu$.

2. **The Ideal Mass:**

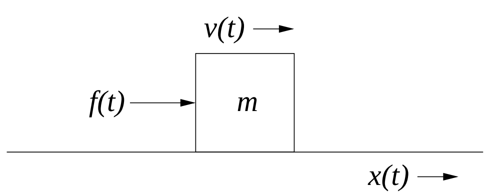

Figure 2: An ideal mass.

- An ideal mass is assumed to move on a friction-less surface or guide-rod.
- The ideal mass is completely rigid.
- By Newton's Second Law: $f(t) = ma(t) = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$

- If we assume the mass moves with a sinusoidal velocity $\tilde{v}(t) = Ce^{j\omega t}$, where C is a complex constant, the corresponding force can be found by differentiation as: $\tilde{f}(t) = mj\omega\tilde{v}(t)$.
- The frequency-domain mass impedance, assuming zero initial conditions, is then given by $Z_m(\omega) = F(\omega)/V(\omega) = mj\omega$.
- The impedance of the mass corresponds to a frequency-dependent force response given a velocity input. At zero frequency it goes to zero (a constant DC velocity implies zero force) and it approaches infinity at high frequencies. The phase is $+\pi/2$: the force always leads velocity by a 1/4 cycle.
- The corresponding mass admittance (force input, velocity output) is $Y_m = 1/(mj\omega)$. A constant DC force produces an infinite velocity output, while a force of infinite frequency results in zero velocity (the mass resists high-frequency motion). The phase is $-\pi/2$: the velocity always lags the applied force by 1/4 cycle.
- The electrical correlate of mechanical mass is an inductor, characterized by $v(t) = L di/dt$. In an analog equivalent circuit, a mass can be represented using an inductor with value $L = m$.

3. **The Ideal Spring:**

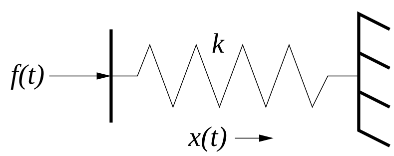

Figure 3: An ideal spring.

- The ideal spring has no mass or internal damping.
- By Hooke's Law: $f(t) = -kx(t) = k\int_0^t v(\tau)d\tau$ for $x(0) = 0$ (valid for small, non-distorting displacements)
- The spring's equilibrium position is given by $x = 0$.
- A positive value of x produces a negative restoring force.
- The spring constant k can also be referred to as the spring stiffness.
- If we assume the spring moves with a sinusoidal velocity $\tilde{v}(t) = Ce^{j\omega t}$, where C is a complex constant, the corresponding force can be found by integration as: $\tilde{f}(t) = k\tilde{v}(t)/(j\omega)$.
- The frequency-domain spring impedance, assuming zero initial conditions, is then given by $Z_k(\omega) = F(\omega)/V(\omega) = k/(j\omega)$.
- The impedance of the spring (frequency-dependent velocity input, force output) at zero frequency approaches infinity (a constant DC velocity produces an infinite restoring force) and it goes to zero at high frequencies (the spring does not resist high-frequency motion). The phase is $-\pi/2$: the restoring force always lags velocity by a 1/4 cycle.
- The corresponding spring admittance (force input, velocity output) is $Y_k = j\omega/k$. A constant DC force implies a zero velocity output, while a force of infinite frequency results in an infinite velocity. The phase is $+\pi/2$: the velocity always leads the restoring force by 1/4 cycle.
- The electrical correlate of a mechanical spring is a capacitor, characterized by $i(t) = C dv/dt$. In an analog equivalent circuit, a spring can be represented by a capacitor of value $C = 1/k$.

4. **Mass / Spring Immittances using the Laplace Transform:**

- The Laplace transform is a convenient tool for analyzing continuous-time systems and solving differential equations, especially when taking into account initial conditions.

- Applying the Laplace transform and the differentiation theorem to the ideal mass: $F(s) = m[sV(s) - v(0)] = m[s^2X(s) - sx_0 - v_0]$, where v_0 and x_0 are initial condition terms for velocity and displacement, respectively.
- Assuming zero initial conditions, the mass impedance is given by $Z_m(s) = F(s)/V(s) = ms$ and mass admittance is $Y_m(s) = V(s)/F(s) = \frac{1}{ms}$.
- The inverse Laplace transform of the admittance,

$$y_m(t) \triangleq \mathcal{L}^{-1}\{Y_m(s)\} = \frac{1}{m}u(t),$$

  where $u(t)$ is the unit step function, can be interpreted as the *impulse response* of the mass given an input $\delta(t)$ of velocity, which corresponds to transferring a unit of momentum to the mass at time 0, with a resulting velocity $v(t) = 1/m$.
- Assuming zero initial conditions ($x_0 = 0$), the spring impedance is given by $Z_k(s) = k/s$ and the spring admittance is $Y_k(s) = s/k$.

1.2 Free Vibrations of Ideal Systems:

In this section, we analyze the behaviour of particular combinations of masses, springs, and dashpots when given an initial displacement and/or velocity. We will consider forced oscillations in a subsequent section. The following linked online simulation provides useful visualizations for this section.

1.2.1 A Mass-Spring Oscillator

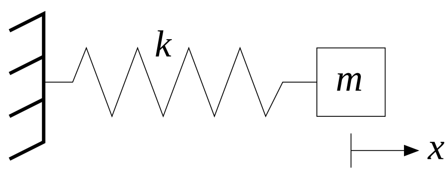

Figure 4: An ideal mass-spring system.

- The series combination of an ideal mass and spring shown in Fig. 4 is characterized by the system equation: $m\ddot{x}(t) + kx = 0$
- This second-order homogeneous differential equation has solutions of the form $x = A\cos(\omega_0 t + \phi)$ representing periodic motion.
- $\omega_0 = \sqrt{k/m}$ is the characteristic (or natural) angular frequency of the system.
- A and ϕ are determined by the initial displacement and velocity.
- There are no losses in the system, so it will oscillate forever.
- This system can be analyzed using the Laplace transform with non-zero initial conditions as follows:

$$m(s^2X(s) - sx_0 - v_0) + kX(s) = 0 \implies X(s) = \frac{sx_0}{s^2 + \omega_o^2} + \frac{v_0}{s^2 + \omega_o^2}$$

  where $\omega_o = \sqrt{k/m}$. From a table of Laplace transform pairs, the solution is found as:

$$x(t) = x_0 \cos(\omega_o t) + \frac{v_0}{\omega_o}\sin(\omega_o t).$$

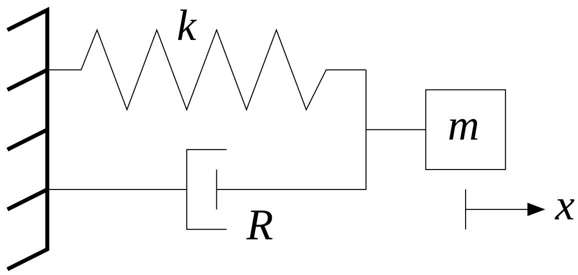

Figure 5: An ideal mass-spring-damper system.

1.2.2 A Damped Mass-Spring Oscillator

A physical system without losses is rare (if not impossible). The introduction of a mechanical dashpot in the mass-spring system provides damping and causes the resulting vibrations to decay over time.

- The series combination of an ideal mass, spring, and damper shown in Fig. 5 is characterized by the system equation: $m\ddot{x} + \mu\dot{x} + kx = 0$
- This second-order homogeneous differential equation has solutions of the form $x = e^{-\alpha t} A\cos(\omega_d t + \phi)$.

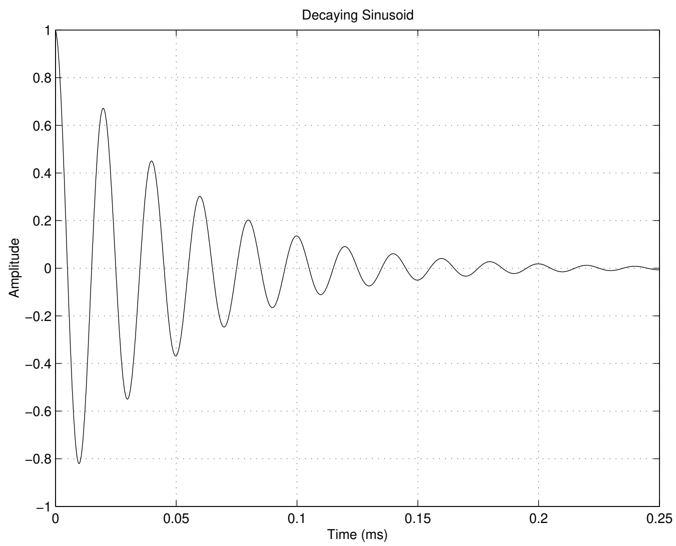

Figure 6: A decaying sinusoid.

- $\alpha = \mu/(2m)$ is a decay constant and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ is the characteristic (or natural) angular frequency of the system.
- A and ϕ are determined by the initial displacement and velocity.
- The natural frequency ω_d is lower than that of the mass-spring system (ω_0).

1.2.3 A One-Mass, Two-Spring System

1. **Longitudinal Motion (along x-axis):**

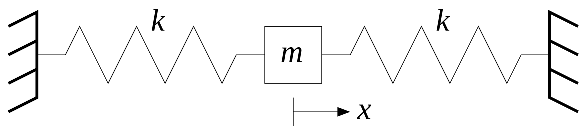

Figure 7: A one-mass, two-spring system: Longitudinal motion.

- The net restoring force on the mass: $f_x(t) = -2kx(t)$
- System natural frequency: $\omega_0 = \sqrt{2k/m}$

1.2.4 A Two-Mass, Three-Spring System

1. **Longitudinal Motion (along x-axis):**

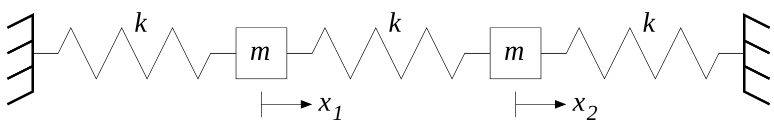

Figure 8: A two-mass, three-spring system: Longitudinal motion.

- System equations:

$$m\frac{d^2x_1}{dt^2} + kx_1 + k(x_1 - x_2) = 0; \quad m\frac{d^2x_2}{dt^2} + kx_2 + k(x_2 - x_1) = 0$$

- Natural frequencies: $\omega = \omega_0, \sqrt{3}\omega_0$, where $\omega_0 = \sqrt{k/m}$

1.2.5 Multiple Mass Systems

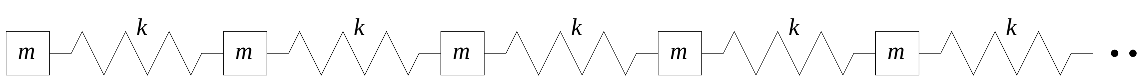

Figure 9: An infinite mass-spring system.

- Each additional mass-spring combination adds another natural mode of vibration per axis of motion.
- As the number of masses and springs increases, the system begins to resemble a uniform string (assuming all masses and all springs are roughly equal in value). Eventually, it becomes more convenient to consider the mass and compression characteristics of the system to be uniformly distributed along its length (as we did in deriving the wave equation).

1.3 Forced Vibrations:

- We are typically interested in the behavior of systems when driven by an external force $f(t)$.
- For a sinusoidal driving force, the resulting response has a transient component and a steady-state term. The transient component, which involves motion at the natural frequency of the system, decays away at a rate proportional to the damping in the system.

- The mechanical impedance, $Z(s) = F(s)/V(s)$, (evaluated at $s = j\omega$) of the system characterizes its steady-state response, after its initial transient behavior has decayed away.
- Components connected in series share a common velocity and their impedances add (i.e., $Z(s) = Z_1(s) + Z_2(s)$).
- Components connected in parallel share a common driving force but can develop different velocities. Admittances add in parallel, while a parallel combination of impedances is given by

$$Z(s) = \frac{1}{\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}} = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}.$$

1.3.1 A Forced, or Driven, Mass-Spring-Damper System

- When a mass-spring-damper system is driven by an external force, the system equation is

$$m\ddot{x} + \mu\dot{x} + kx = f(t)$$

- The driven mass-spring-damper system can be described using the Laplace Transform as (assuming $v(0) = 0$ and $x(0) = 0$):

$$m\ddot{x} + \mu\dot{x} + kx = f(t) \quad \Longleftrightarrow \quad V(s)\,[ms + \mu + k/s] = F(s),$$

and its *impedance* determined as

$$Z(s) \stackrel{\Delta}{=} F(s)/V(s) = ms + \mu + k/s.$$

This impedance expression, $Z(s)$, can be evaluating for $s = j\omega$, as shown in Fig. 10 for three different damping constants.

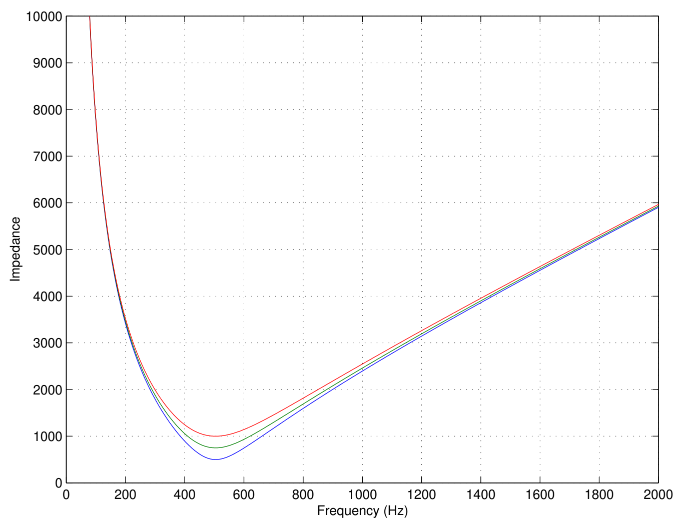

Figure 10: Impedance of a mass-spring-damper system.

- Mechanical driving-point impedance minima indicate the frequencies at which a sinusoidal driving force of constant amplitude will produce the greatest mass velocity (and displacement).
- The Laplace Transform for the mass-spring-damper system can also be expressed in terms of its displacement (assuming $x(0) = 0$ and $v(0) = 0$):

$$m\ddot{x} + \mu\dot{x} + kx = f(t) \quad \Longleftrightarrow \quad X(s)\left[ms^2 + \mu s + k\right] = F(s)$$

The steady-state force-to-displacement response is then given by:

$$X(s)/F(s) = \frac{1}{ms^2 + \mu s + k},$$

and is plotted in Fig. 11 for three damping constants. *Resonance* occurs when a system is driven near its natural frequency.

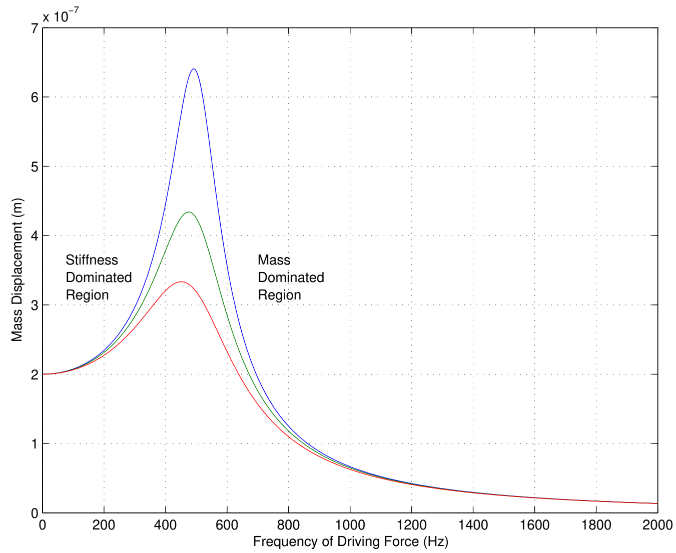

Figure 11: Mass displacement vs. driving frequency for mass-spring-damper system.

- Below resonance, the system is said to be "stiffness dominated".
- Above resonance, the system is said to be "mass dominated" and the resulting displacement approaches zero with increasing frequency.

2 Discretization Methods:

There are a variety of approaches to solving continuous-time system equations using discrete-time methods. In general, a higher *sampling rate* will produce more accurate results. In this section, we overview several methods, including Finite Differences and the Bilinear Transform.

2.1 Backward Finite Differences

The Finite Difference technique replaces derivative expressions in differential equations with discrete-time difference approximations.

- The backward finite difference (FD) approximation is given by:

$$\frac{d}{dt}x(t) \triangleq \lim_{\delta \to 0} \frac{x(t) - x(t-\delta)}{\delta} \approx \frac{x[n] - x[n-1]}{T}$$

  where T is the sampling period.

- In the frequency-domain, the FD approximation is defined by the mapping:

$$s \leftarrow \frac{1 - z^{-1}}{T},$$

  where s is the Laplace Transform frequency variable and z is the z-Transform frequency variable.

- The inverse finite difference substitution is given by:

$$z = \frac{1}{1 - sT} = 1 + sT + (sT)^2 + \ldots,$$

- The FD approximation maps analog dc ($s = 0$) to digital dc ($z = 1$).
- By noting that the FD approximation maps an infinite analog frequency ($s = \infty$) to $z = 0$, it should be clear that non-zero poles and zeros are warped in potentially undesireable ways.
- The FD approximation does not alias because the conformal mapping $s = 1 - z^{-1}$ is one to one.
- By applying the FD twice, the second derivative is found as:

$$\frac{d^2}{dt^2}x(t) \approx \frac{x[n] - 2x[n-1] + x[n-2]}{T^2}$$

- The Matlab example, msd_fd.m, demonstrates the use of the finite difference approach to simulate the motion of the mass-spring-damper system.

2.2 Centered Finite Differences

- Assuming that one sample of "look-ahead" is available, another finite difference scheme can be defined as:

$$\frac{d}{dt}x(t) \approx \frac{x[n+1] - x[n-1]}{2T}$$

  and

$$\frac{d^2}{dt^2}x(t) \approx \frac{x[n+1] - 2x[n] + x[n-1]}{T^2}$$

- These equations represent zero phase filters, producing no delay at any frequency.
- Note, however, that the second derivative approximation is not formed by applying the first derivative approximation to itself. Thus, the s- to z-plane mapping is different for the two approximations. Nor is it straight-forward to evaluate the s- to z-plane mapping in either case.
- For the first derivative approximation, we can find:

$$z = sT \pm \sqrt{1 + (sT)^2}.$$

- For frequencies above $|\Omega T| > 1$ (or frequencies above $f_s/2\pi$), the resulting values of z are outside the unit circle, and thus unstable.
- Thus, while the centered finite difference approximation has desirable qualities, it is limited by frequency range considerations.

2.3 The Bilinear Transform

- The bilinear transform is an algebraic transformation between the continuous-time and discrete-time frequency variables s and z, respectively. It is therefore appropriate only when a closed-form filter representation in s exists.
- The bilinear transform is defined by the substitution:

$$s = c\frac{1 - z^{-1}}{1 + z^{-1}}, \quad c > 0,$$

  where T is the sampling period.

- The bilinear transform maps the entire continuous-time frequency space, $-\infty \leq \Omega \leq \infty$, onto the discrete-time frequency space $-\pi \leq \omega \leq \pi$. Continuous-time dc ($\Omega = 0$) maps to discrete-time dc ($\omega = 0$) and infinite continuous-time frequency ($\Omega = \infty$) to the *Nyquist frequency* ($\omega = \pi$). Thus, a nonlinear *warping* of the frequency axes occurs.

- Since the $j\Omega$-axis in the s plane is mapped exactly once around the unit circle of the z plane, no aliasing occurs.

- The constant c is typically given by $c = 2/T$. This parameter provides one degree of freedom for the mapping of a particular finite continuous-time frequency to a particular location on the z-plane unit circle. The warping of frequencies is given for the discrete-time frequency in terms of c and Ω as

$$\omega = 2 \arctan(\Omega/c).$$

- If one wishes to map a specific "analog" frequency (Ω) to the equivalent "digital" frequency (ω), the constant c is found as

$$c = \frac{\Omega}{\tan(\omega/2)}.$$

- Because of the nonlinear compression of the frequency axis, use of the bilinear transformation should be limited to filters with a single transition band (lowpass and highpass) or resonance (narrow bandpass and bandstop).

- The Matlab example, msd_fdbt.m, compares the results of using both the finite difference and bilinear transform approaches to simulate the motion of a mass-spring-damper system.

2.4 Finite-Differences vs. the Bilinear Transform

- The FD approximation is given by the mapping:

$$z = \frac{1}{1 - sT}.$$

- The Bilinear transform is given by the mapping:

$$z = \frac{1 + s/c}{1 - s/c}.$$

- The s- to z-plane mapping for both the finite difference and bilinear transform is shown in Fig. 2.4. Note that the bilinear transform maps the $j\omega$ axis exactly onto the unit circle.

- Both methods preserve order, stability, and are free from aliasing. Both methods provide an ideal frequency mapping at zero frequency but compressively warp higher frequencies.

- Only the finite difference method introduces artificial damping at higher frequencies. Because of this, it is even possible that unstable s-plane poles could be mapped to stable z-plane poles.

2.5 Other Numerical Methods

Numerous other methods exist for solving systems of partial differential equations. The interested reader could start with the following general overview of finite difference schemes.

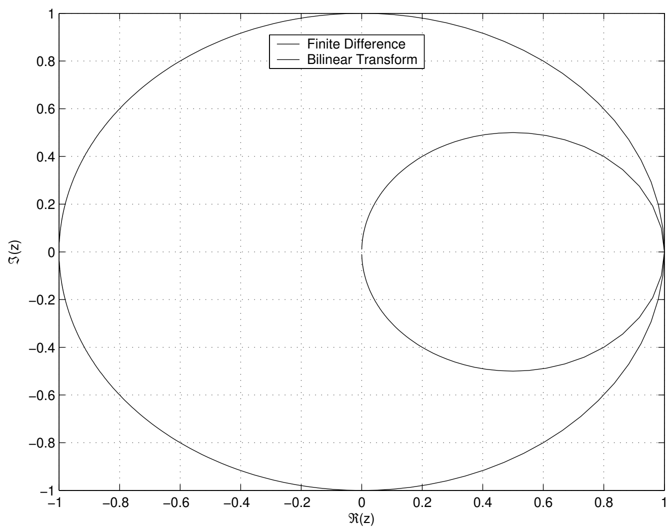

Figure 12: Finite Difference and Bilinear Transform $j\omega$ to z-plane frequency mapping.