MUMT 618: Week #8

1 Acoustic Systems

1.1 Acoustic Variables

- When analyzing acoustic systems, the primary variables of interest are pressure and velocity.
- Acoustic pressure is defined as force per unit area. It can also be interpreted as energy density.
- In acoustic systems, velocity can be expressed either in terms of *particle velocity* (v) or *volume velocity* (u).
- In enclosed air ducts, it is more typical to consider *volume velocity* or particle velocity times the cross-section area A of the duct ($u = Av$).

1.2 Acoustic Wave Motion

- Because gases, such as air, do not support shear stresses (to first approximation), wave motion in air is primarily longitudinal.
- Most acoustic music instruments have air columns that are roughly cylindrical or conical in shape.
- When analyzing the acoustic behavior of these systems, we are most often interested in one-dimensional wave motion along the principal axis of the air column.
- Longitudinal wave motion along the principal axis of cylinders is planar.
- Longitudinal wave motion along the principal axis of cones is spherical.

1.3 Acoustic Impedance

- Acoustic impedance is defined as $Z(f) = P(f)/U(f)$, where $P(f)$ and $U(f)$ are sinusoidal quantities of pressure and volume flow, respectively.
- An air column's *input impedance* is defined as its sinusoidal pressure response to a driving sinusoidal volume velocity source at the input to the air column.
- An air column's *impulse response* is defined as its time-domain pressure response to the application of a volume velocity unit impulse at the input of the system.
- The impulse response and input impedance of an air column are time- and frequency-domain correlates, as shown in Fig. 1 below.
- The input impedance peaks indicate the frequencies at which a volume velocity source will produce the greatest pressure variations at the input to the air column.
- The input impedance minima indicate the frequencies at which a pressure source will produce the greatest volume velocity variations at the input to the air column.

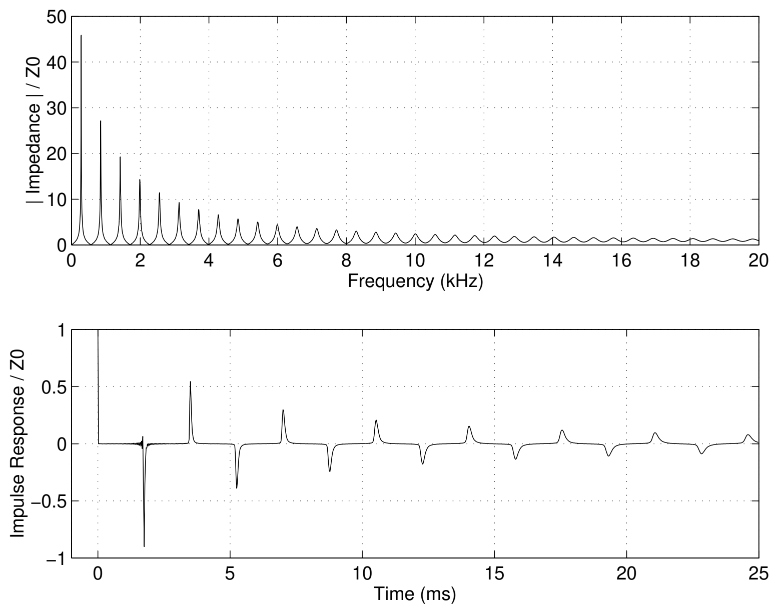


Figure 1: Theoretical input impedance magnitude (top) and impulse response (bottom) of a cylindrical bore closed at the input and open at its output end, normalized by the bore characteristic impedance.

- For pressure controlled driving mechanisms (such as the brass player's lips), the input impedance peaks indicate the frequencies at which air column vibrations will cooperate with the driving mechanism to sustain steady oscillations.
- In the past, input impedance was measured using a variable-frequency volume velocity source of constant amplitude. However, time-domain and multi-microphone measurement techniques have become popular during recent years.

1.4 Newton's Second Law for Plane Waves

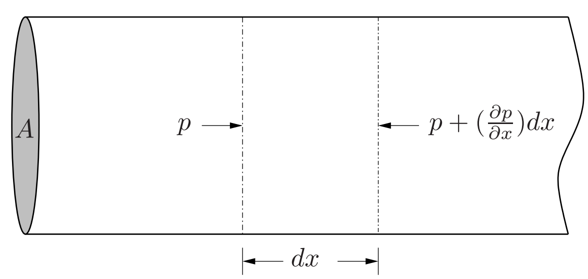


Figure 2: An infinitesimal section of an enclosed fluid with pressure variation.

- We can evaluate Newton's Second Law for the infinitesimal section of fluid diagrammed in Fig. 2, which is assumed to be subjected to a differential pressure along its length.
- If the fluid has a mass density ρ and cross-sectional area A, the section mass is given by $\rho A dx$.
- From Newton's Second Law ($F = ma$), we then have:

$$F = A\left[p - (p + (\frac{\partial p}{\partial x})dx)\right] = -Adx\frac{\partial p}{\partial x}$$

$$= \rho A dx \frac{\partial v}{\partial t},$$

where v is particle velocity.

- From this expression, we find:

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial v}{\partial t}. \tag{1}$$

- Converting to an expression for volume velocity ($u = Av$), we find:

$$\frac{\partial p}{\partial x} = -\frac{\rho}{A} \frac{\partial u}{\partial t}. \tag{2}$$

1.5 Hooke's Law for Plane Waves

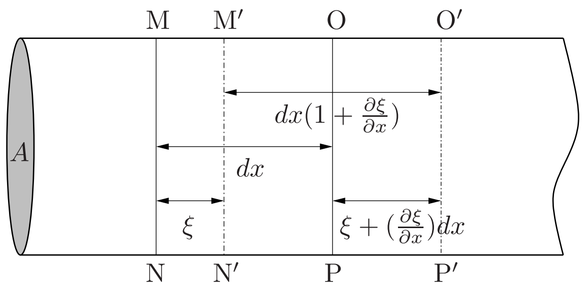


Figure 3: An infinitesimal section of an enclosed fluid with volumetric dilation.

- Consider the infinitesimal section of fluid MNOP with cross-sectional area A diagrammed in Fig. 3. Under the influence of a pressure change p, the section is displaced to points M′N′O′P′, undergoing a volume dilation in the process.

- Assuming section MNOP has a volume Q, the volume of M′N′O′P′ is given by

$$Q + dQ = A dx \left(1 + \frac{\partial \xi}{\partial x}\right). \tag{3}$$

- For small dilation, Hooke's law for fluids provides an accurate approximation to the relationship between the change in applied pressure, the resulting strain, and the bulk modulus K,

$$p = -K \frac{dQ}{Q} = -K \frac{\partial \xi}{\partial x}. \tag{4}$$

- The temperature within sound waves tends to rise where the air is compressed and fall where it is expanded. Because the wavelengths of sound are generally large, the pressure maxima and minima are thus spread far apart and the temperature gradients are too small to produce significant heat conduction within the medium. This behaviour is referred to as an *adiabatic* process.

- The adiabatic bulk modulus for sound waves can be expressed in terms of the ratio of specific heats of air at constant pressure and constant volume $\gamma = C_p/C_v = 1.4$ and the average atmospheric pressure P_0, or the density of air ρ and the speed of sound c as $K_{ad} = \gamma P_0 = \rho c^2$.

- The speed of sound does not vary with atmospheric pressure but it does vary with temperature. For air at temperature ΔT in degrees Celsius,

$$c \approx 332(1 + 0.00166\Delta T)\,\mathrm{m\,s^{-1}}. \tag{5}$$

1.6 The Wave Equation for Plane Waves

- If we differentiate Eq. (1) once with respect to x and differentiate Eq. (4) twice with respect to time t, noting that particle velocity $v = \partial\xi/\partial t$, we find

$$\frac{\partial^2 p}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 p}{\partial x^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \tag{6}$$

  which is the one-dimensional wave equation for lossless pressure plane waves.

1.7 Lumped Acoustic Elements

The fundamental wavelength of sound produced by a musical instrument is generally much larger than the dimensions of certain of its component parts, such as toneholes and mouthpieces. The behavior of large wavelength sound waves within small structures is generally well approximated by assuming uniform pressure throughout the volume of interest [Fletcher and Rossing, 1991]. In this way, we can analyze small acoustic components in terms of ideal lumped elements.

1.7.1 An Acoustic Mass (Inertance)

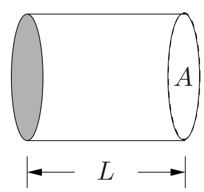


Figure 4: A short acoustic tube.

- In the low-frequency limit, the air within a short, open tube will be displaced by equal amounts at both its ends when subjected to an external pressure at one end only. If the length and cross section of the tube are given by L and A, respectively, then the enclosed air has a mass of ρLA, where ρ is the mass density of air.

- The acoustic response of this system is analyzed by assuming an applied sinusoidal pressure of the form $P(\omega)$. Using Newton's second law (force = mass × acceleration),

$$P(\omega)A = \rho LA\,\frac{dV(\omega)}{dt} = \rho L\,\frac{dU(\omega)}{dt}, \tag{7}$$

  where $U(\omega) = AV(\omega)$ is the acoustic volume velocity of the air mass.

- The volume velocity response to the applied pressure will also vary sinusoidally with frequency ω, so that Eq. (7) reduces to

$$P(\omega) = \frac{j\omega\rho L}{A} U(\omega). \tag{8}$$

- The acoustic impedance of the tube is then given by

$$Z(\omega) = \frac{P(\omega)}{U(\omega)} = \left(\frac{\rho L}{A}\right) j\omega \tag{9}$$

  (or $Z(s) = (\rho L/A)s$ when expressed in terms of the Laplace transform), where initial conditions are assumed equal to zero.

- In the low-frequency limit, the open tube is called an acoustic inductance or an *inertance* and it has a direct analogy to the inductance in electrical circuit analysis or the mass in mechanical system analysis.
- The impedance of a mechanical mass is equal to ms, and thus the open tube has an equivalent acoustic "mass" equal to $\rho L/A$. An inertance is also sometimes referred to as a *constriction* [Morse, 1981, p. 234].

1.7.2 An Acoustic Spring (Cavity)

- The acoustic analog of the electrical capacitor or the mechanical spring is a cavity, or a *tank* [Morse, 1981, p. 234].
- In the low-frequency limit, an applied external pressure will compress the enclosed air, which then acts like a spring because of its elasticity. Assuming a cavity volume Q, an increase in applied pressure P will decrease this volume by an amount $-dQ$.
- The ratio of change in volume to original volume is called volume strain or dilation and is given by $\epsilon = dQ/Q$. Using Hooke's law, we find

$$P = -K\frac{dQ}{Q} = -K\epsilon, \tag{10}$$

  where K is the bulk modulus as discussed earlier.
- Using Eq. (10) and writing the change in cavity volume, $-dQ$, in terms of the sinusoidal volume velocity U as

$$-dQ = \int U dt = \frac{U}{j\omega}, \tag{11}$$

  the acoustic impedance of the cavity is given in terms of the Laplace transform by

$$Z(s) = \left(\frac{\rho c^2}{Q}\right)\frac{1}{s}. \tag{12}$$

- The impedance of a mechanical spring is equal to k/s, so by analogy the acoustic cavity has an equivalent "spring constant" equal to $\rho c^2/Q$.

1.7.3 The Helmholtz Resonator

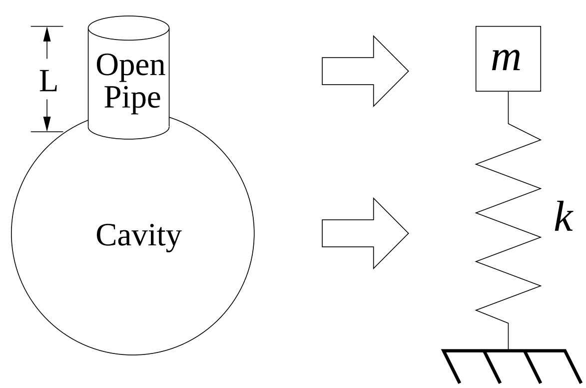


Figure 5: The Helmholtz resonator and its mechanical correlate.

- In the "low-frequency limit", an open tube is a direct acoustic correlate to the mechanical mass.

- In the “low-frequency limit”, a cavity is a direct acoustic correlate to the mechanical spring.
- Using Newton’s Second Law to model the air mass in the tube and Hooke’s Law for fluids to model the compressibility of the air cavity, a sinusoidal solution can be found with natural frequency $\omega_0 = c\sqrt{A/(LQ)}$, where c is the speed of sound in air, A is the cross-sectional area of the tube, L is the length of the tube, and Q is the volume of the cavity.

2 Cylindrical Air Column Acoustics

The resonator of a musical instrument serves to emphasize certain desirable frequencies of sound, corresponding to its normal modes. A wind instrument air column is a subclass of general resonators and functions to encourage the production of sustained, quasi-periodic oscillations of sound within a feedback system. In order that the air column be musically useful, it is necessary that it support sound production over a wide range of frequencies. This is accomplished by allowing the tube length to be variable while also encouraging sound production based on higher normal modes. Acoustically speaking, two principle requirements need be met for a wind instrument resonator to be musically useful. It is first necessary that the ratio between the first and successively higher normal modes be independent of horn length. In this way, the perceived *timbre* or spectral content of sounds will remain similar over the full range of the bore. Further, this permits the production of sounds based on second and third resonances, or *overblowing*, over the full length of the tube. Secondly, a stable regenerative process associated with the nonlinear excitation mechanism in wind instruments is favored when the successive partial frequencies, particularly the first few, are related to the fundamental frequency by integer multiple ratios [Benade, 1959, 1977].

In wind instrument bores, the primary mode of wave propagation is along the central axis of the tube. Equations describing this wave motion are possible if a coordinate system can be found in which one coordinate surface coincides with the walls of the given pipe and in which the wave equation is separable [Fletcher and Rossing, 1991, p. 187]. There are 11 coordinate systems in which the Helmholtz equation is separable. One-parameter waves, however, are possible only in rectangular, circular cylindrical, and spherical coordinates, which correspond to pipes of uniform cross-section and conical horns, respectively [Putland, 1993]. Wave propagation in the other separable coordinate systems must be comprised of an admixture of orthogonal modes and be a function of more than one coordinate. In order to perfectly meet the requirements of a musically useful wind instrument bore as discussed above, one-parameter wave propagation is necessary. Thus, cylindrical pipes and conical horns are the two obvious choices for wind instrument bores. All wind instrument air columns are based on shapes roughly corresponding to cylinders or cones. Further, accurate representations of wave propagation in actual, imperfectly shaped bores can be well approximated in terms of cylindrical and conic sections.

2.1 Cylindrical Pipes: Modes of Propagation

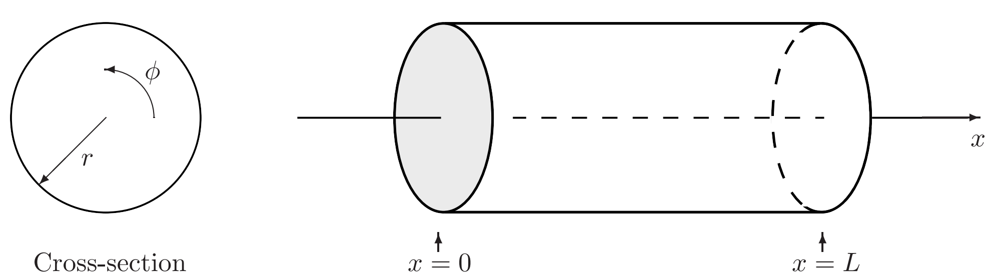


Figure 6: A cylindrical pipe in cylindrical polar coordinates.

- A short section of a cylindrical pipe in cylindrical polar coordinates (r, ϕ, x) is depicted in Fig. 6.

- Longitudinal wave motion is possible along the principal axis, as well as in planes orthogonal to the principal axis. However, it will be seen that these *transverse* modes are only weakly excited in musical instrument bores of small diameter.

- The wave equation in this geometric coordinate system is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{13}$$

- Assuming sinusoidal solutions, a complete general solution to the wave equation in circular polar coordinates is given by

$$P_{mn}(r, \phi, x) = \Phi(\phi)R(r)X(x) = P_0 \cos(m\phi) J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{-jk_{mn}x}, \tag{14}$$

  where J_m is a Bessel function, a is the radius of the cylinder, and α_{mn} denotes the positive zeros of the derivative $J'_m(\alpha_{mn})$.

- The wavenumber for a sinusoidal disturbance propagating axially along the tube, k_{mn}, varies with mode (m, n) as

$$k_{mn}^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\alpha_{mn}}{a}\right)^2, \tag{15}$$

  where mode paramter m refers to nodal diameters and n refers to nodal circles (see Fig. 7 below).

- One-dimensional plane-wave propagation corresponds to mode $(0, 0)$, for which $k_{00} = k = \omega/c$.

- Higher modes will propagate only if k_{mn}^2 is positive (or k_{mn} is real), so that the frequency must exceed a critical (often called "cutoff") value given by

$$\omega_c = \frac{\alpha_{mn} c}{a}. \tag{16}$$

- For frequencies less than ω_c, the mode is evanescent and decays exponentially with distance (see animation).

- The plane-wave mode has a cut-off frequency of zero and no transverse wave motion.

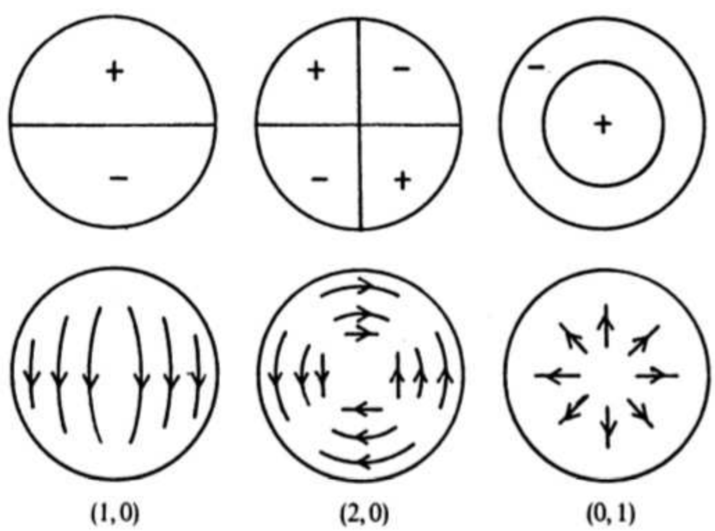


Figure 7: Pressure (upper) and flow (lower) patterns for the lowest three higher-order modes in a cylindrical pipe (from Fletcher and Rossing [1991]).

- The next two propagating modes are the $(1,0)$ and $(2,0)$ nodal plane modes, which have cutoff frequencies $\omega_c = 1.84c/a$ and $\omega_c = 3.05c/a$, respectively.
- A typical clarinet has a radius of about 7.5 millimeters for a majority of its length, while that of a flute is about 8.5 millimeters. With the speed of sound approximated by $c = 347.23$ meters per second for a temperature of 26.85°C, these cutoff frequencies are 13.56 kHz and 22.5 kHz for the clarinet and 11.96 kHz and 19.8 kHz for the flute.
- The first propagating transverse mode is well within the range of human hearing. However, excitation of this mode requires transverse circular motion, which will not occur with any significance in musical instruments.
- Evanescent mode losses may be possible in turbulent regions, such as in the mouthpiece and near toneholes.

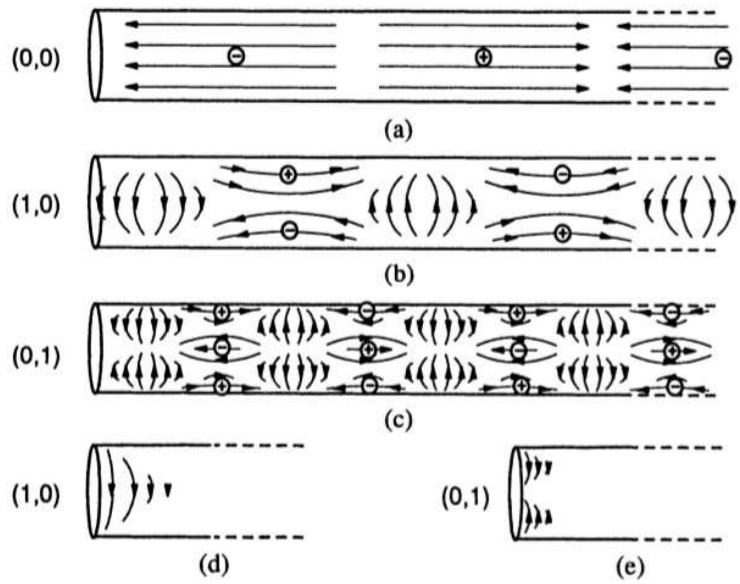


Figure 8: Acoustic flow patterns and pressure maxima and minima for modes in a cylindrical duct. The lowest two systems represent evanescent, non-propagating mode patterns (from Fletcher and Rossing [1991]).

2.2 Cylindrical Pipes: Plane Wave Propagation

- Given the previous analysis, wave motion in cylindrical woodwind bores is primarily planar and along the principal axis of the air column. The equation of motion for a pressure wave propagating in this way along the x-axis with sinusoidal time dependence has the form

$$P(x,t) = C^{+} e^{j(\omega t - kx)}, \tag{17}$$

  which is a solution to the one-dimensional wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{18}$$

- Plane waves of sound can theoretically propagate without reflection or loss along the principal axis of an infinite cylindrical pipe, assuming the walls are rigid, perfectly smooth, and thermally insulating.
- From Newton's law, pressure and volume velocity in a cylindrical pipe are related by

$$\frac{\partial p}{\partial x} = -\frac{\rho}{A}\frac{\partial u}{\partial t}, \tag{19}$$

  where A is the cross-sectional area of the pipe and ρ is the mass density of air.

- For pressure waves given by Eq. (17), the associated volume flow is found from Eq. (19) as

$$U(x,t) = \left(\frac{A}{\rho c}\right) C^+ e^{j(\omega t - kx)} \tag{20}$$

and the characteristic or wave impedance is

$$Z_c = \frac{P(x,t)}{U(x,t)} = \frac{\rho c}{A}. \tag{21}$$

- Thus, *traveling-wave* components of pressure and velocity are in-phase and related by a purely resistive wave impedance.

2.3 Finite Length Cylindrical Pipes

- In a pipe of finite length, propagating wave components will experience discontinuities at both ends.
- A longitudinal wave component which encounters a discontinuous and finite load impedance Z_L at one end of the tube will be partly reflected back into the tube and partly transmitted into the discontinuous medium.
- Wave variables in a finite length tube are then composed of superposed right- and left-going traveling waves. In this way, sinusoidal pressure in the pipe at position x is given by

$$P(x,t) = \left[C^+ e^{-jkx} + C^- e^{jkx}\right] e^{j\omega t}, \tag{22}$$

where C^+ and C^- are complex amplitudes.

- From Eq. (19), the corresponding volume velocity is found to be

$$\begin{aligned} U(x,t) &= \left(\frac{A}{\rho c}\right) \left[C^+ e^{-jkx} - C^- e^{jkx}\right] e^{j\omega t} \\ &= \frac{1}{Z_c} \left[C^+ e^{-jkx} - C^- e^{jkx}\right] e^{j\omega t} \end{aligned} \tag{23}$$

- At any particular position x and time t, the pressure and volume velocity traveling-wave components are related by

$$P^+ = Z_c U^+ \qquad P^- = -Z_c U^- \tag{24}$$

with

$$P = P^+ + P^- \qquad U = U^+ + U^- \tag{25}$$

- The plus (+) superscripts indicate wave components traveling in the positive x-direction or to the right, while negative (−) superscripts indicate travel in the negative x-direction or to the left.
- The characteristic wave impedance Z_c is a frequency-domain parameter, though for plane waves of sound it is purely real and independent of position. Therefore, these relationships are equally valid for both frequency- and time-domain analyses of pressure and volume velocity traveling-wave components.
- Traveling waves of sound are typically reflected at an acoustic discontinuity in a frequency-dependent manner.
- A frequency-dependent reflection coefficient, or *reflectance*, characterizes this behavior and indicates the ratio of incident to reflected complex amplitudes at a particular frequency.
- Similarly, the ratio of incident to transmitted complex amplitudes at a particular frequency is characterized by a frequency-dependent transmission coefficient, or *transmittance*.

- For a pipe which extends from $x = 0$ to $x = L$ and is terminated at $x = L$ by the load impedance Z_L, the pressure wave reflectance is

$$\frac{C^-}{C^+} = e^{-2jkL}\left[\frac{Z_L - Z_c}{Z_L + Z_c}\right] \tag{26}$$

  and the transmittance is

$$\frac{P(L,t)}{C^+} = e^{-jkL}\left[\frac{2Z_L}{Z_L + Z_c}\right]e^{j\omega t}. \tag{27}$$

- The phase shift term e^{-2jkL} in Eq. (26) appears as a result of wave propagation from $x = 0$ to $x = L$ and back and has unity magnitude.

- The load impedance Z_L characterizes sound reflection and radiation at the end of the pipe.

- For low-frequency sound waves, the open end of a tube can be approximated by $Z_L = 0$. In this limit, the bracketed term of the reflectance becomes negative one, indicating that pressure traveling-wave components are reflected from the open end of a cylindrical tube with an inversion (or a 180° phase shift). There is no transmission of incident pressure into the new medium when $Z_L = 0$.

- If the pipe is rigidly terminated at $x = L$, an appropriate load impedance approximation is $Z_L = \infty$, corresponding to $U(L,t) = 0$ for all time. The bracketed term in Eq. (26) is then equal to one, which implies that pressure traveling waves reflect from a rigid barrier with no phase shift and no attenuation. The pressure "transmittance" (a bit of a misnomer in this case), has a magnitude of two at the rigid barrier.

- The impedance at $x = 0$, or the *input impedance* of the cylindrical tube, is given by

$$Z_{in} = \frac{P(0,t)}{U(0,t)} = Z_c\left[\frac{C^+ + C^-}{C^+ - C^-}\right] \tag{28}$$

$$= Z_c\left[\frac{Z_L\cos(kL) + jZ_c\sin(kL)}{jZ_L\sin(kL) + Z_c\cos(kL)}\right]. \tag{29}$$

- The input impedance of finite length bores can be estimated using the low-frequency approximation $Z_L = 0$ for an open end and $Z_L = \infty$ for a closed end. In this case, Equation (29) reduces to

$$Z_{in} = jZ_c\tan(kL) \tag{30}$$

  for the ideally open pipe and

$$Z_{in} = -jZ_c\cot(kL) \tag{31}$$

  for the rigidly terminated pipe.

- In the low-frequency limit, $\tan(kL)$ is approximated by kL and the input impedance of the open pipe reduces to $j\omega\rho L/A$. This is the expression for the impedance of a short open tube, or an acoustic inertance.

- Making a similar approximation for $\cot(kL)$, the input impedance of the rigidly terminated pipe reduces to $-(j/\omega)(\rho c^2/LA)$, which is equivalent to the impedance of a cavity in the low-frequency limit.

- By equating an open pipe end at $x = 0$ with a value of $Z_{in} = 0$ in the previous expressions, the resonance frequencies of the open-closed (o-c) pipe and the open-open (o-o) pipe are given for $n = 1, 2, \ldots$ by

$$f^{(\text{o-c})} = \frac{(2n-1)c}{4L} \tag{32}$$

$$f^{(\text{o-o})} = \frac{nc}{2L}, \tag{33}$$

  respectively.

- The open-closed pipe is seen to have a fundamental wavelength equal to four times its length and higher natural frequencies that occur at odd integer multiples of the fundamental frequency. The following link provides some animations of longitudinal standing-wave patterns in pipes.
- The open-open pipe has a fundamental wavelength equal to two times its length and higher natural frequencies that occur at all integer multiples of the fundamental frequency.
- The input impedance of an acoustic structure provides valuable information regarding its natural modes of vibration. Various methods for measuring and/or calculating the input impedance of musical instrument bores have been reported [Benade, 1959, Backus, 1974, Plitnik and Strong, 1979, Caussé et al., 1984].

3 Cylindrical Air Column Modeling

The acoustic behavior of wind-instrument air columns has traditionally been described in the frequency-domain, where such characteristics as normal-mode frequencies and decay rates and tonehole-lattice cutoff frequencies are easily identifiable. The requirements for wind-instrument bores are also typically stated in terms of frequency-domain constraints. Research on wind-instrument nonlinear excitation mechanisms and their complex coupling with the air column, as well as interest in the simulation of a complete wind instrument, however, has led to the development of time-domain models of wind-instrument sound production.
The linear time-domain response of an air column is represented by either its impulse response $h(t)$ or its reflection function $r(t)$. The impulse response is the pressure response at the bore entrance to the introduction of a unit volume velocity impulse. Because the entryway is represented by a closed end after the impulse is introduced, $h(t)$ is a slowly decaying function. The reflection function is the pressure response at the entrance of the air column to the introduction of a unit pressure impulse. There are no reflections at the bore entryway for this response function, so that it decays to zero much more quickly than the impulse response. Traditional computational methods for the simulation of time-domain pressure propagation in a complete wind instrument involve the convolution of mouthpiece pressure with an air column response function during each time sample period. In this respect, $r(t)$ serves to greatly reduce the number of mathematical operations necessary for calculation of the model output. Digital waveguide modeling of wave propagation in wind-instrument air columns incorporates the advantages of the reflection function approach, and further reduces computational requirements by "pulling out" the delay inherent in $r(t)$ and implementing it in the form of digital delay lines.

3.1 An Ideal Digital Waveguide Model

- In a previous section, it was found that wave motion in wind instruments with cylindrical bores is adequately represented by plane-wave propagation along the length of the pipe.
- The one-dimensional wave equation for plane waves,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}\,, \tag{34}$$

  accurately represents lossless wave propagation along the central axis of a cylindrical tube, subject to the boundary conditions at both its ends.
- A discrete-time and -space traveling-wave simulation of this wave propagation is then given by

$$p\left(t_n, x_m\right) = p^{+}(n-m) + p^{-}(n+m), \tag{35}$$

  where the spatial sampling distance is given by $X = cT_s$ and the time sampling interval is $T_s = 1/f_s$.
- At the open end of a pipe, the pressure can be (ideally) approximated as zero for low-frequency wave components.

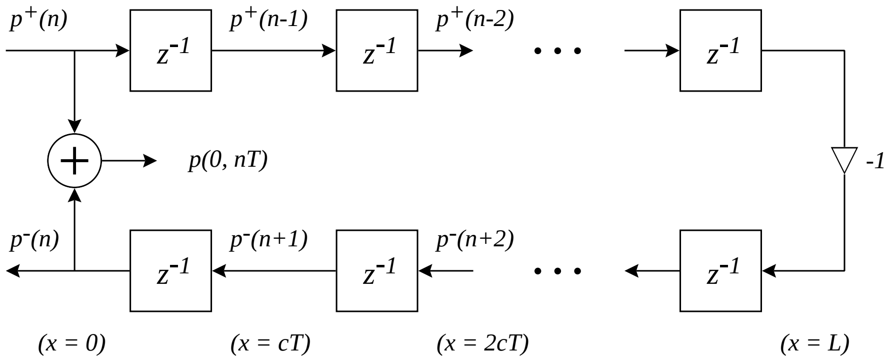


Figure 9: Digital waveguide implementation of ideal, lossless plane-wave propagation in a cylindrical tube. The z^{-1} units represent one-sample delays.

- Figure 9 represents the digital waveguide implementation of lossless plane-wave pressure propagation in an ideally terminated cylindrical tube.
- The negative one multiplier at $x = L$ implements the low-frequency, open-end approximation for traveling-wave pressure reflection.
- The system of Fig. 9 is a discrete-time and -space implementation of an ideal cylindrical bore reflection function.
- The continuous-time reflectance seen from the entrance of a cylindrical tube of length L is given by

$$\mathcal{R}(\omega) = e^{-2jkL} \left[\frac{Z_L(\omega) - Z_c}{Z_L(\omega) + Z_c} \right], \tag{36}$$

  where Z_c is the real characteristic wave impedance of the cylindrical pipe, and $Z_L(\omega)$ is the load impedance at $x = L$.
- For $Z_L(\omega) = 0$, which corresponds to zero pressure at $x = L$, $R(\omega) = -e^{-2jkL}$. The resulting continuous-time reflection function $r(t) = -\delta(t - 2L/c)$ is found as the inverse Fourier transform of $\mathcal{R}(\omega)$ when viscothermal losses are ignored.
- This time delay is realized in the digital domain by the delay lines of Fig. 9. The bracketed term in Eq. 36 represents the reflection characteristic at the end of the pipe and must be properly discretized to account for more realistic boundary conditions.

3.2 Model Generalizations

- The open end of a cylindrical pipe is poorly represented by a load impedance $Z_L(\omega) = 0$.
- In general, the acoustic properties of the end of the wind-instrument bore are characterized by a non-zero and complex valued load impedance and this impedance can be expressed as a lumped traveling-wave reflectance, as given by the bracketed term of Eq. (36).
- Figure 10 represents the digital waveguide implementation of plane-wave pressure propagation in a cylindrical tube terminated by the reflectance $\mathcal{R}_L(z)$. Thermal and viscous boundary layer losses are neglected in this model.
- The waveguide structure of Fig. 10 can be further simplified by limiting observation of physical pressure to the input of the tube.
- Then, by linearity and time-invariance, the digital filter representing $\mathcal{R}_L(z)$ can be “pushed” through the lower delay line to its output and a single delay line used for the simulation, as shown in Fig. 11.

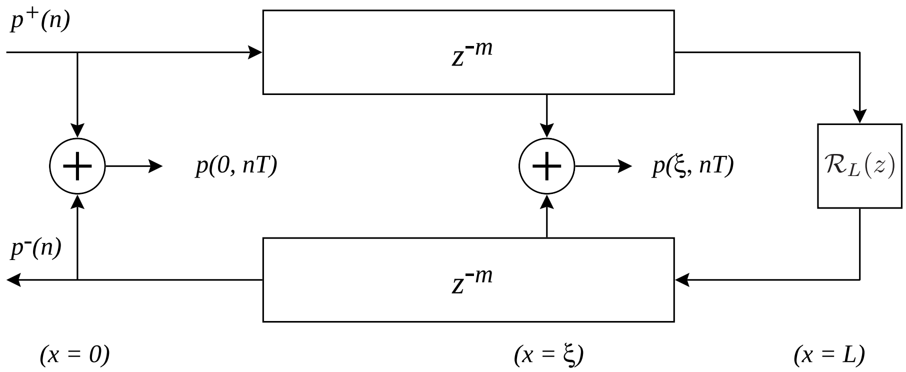


Figure 10: Digital waveguide implementation of plane-wave propagation in a cylindrical tube, neglecting viscothermal losses.

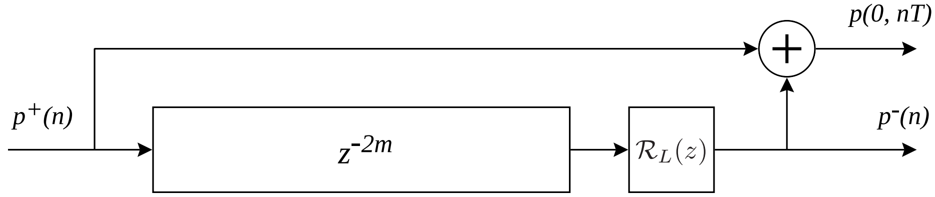


Figure 11: Simplified digital waveguide implementation of plane-wave propagation in a cylindrical tube using a single delay line and neglecting viscothermal losses. The pressure observation point is constrained to the entryway of the bore.

3.3 Impulse Response

- The impulse response $h(t)$ of a wind-instrument air column, defined as the inverse Fourier transform of its input impedance, implies an acoustic feedback loop because of the inherent closure of the bore entryway after the introduction of a volume velocity unit impulse.
- The infinite termination impedance of the closed end results in a non-inverting reflection of pressure, so that the waveguide model of a closed-open cylindrical bore is as shown in Fig. 12.
- The waveguide "input impedance" is found by calculating the system impulse response and transforming it to the frequency domain using the discrete Fourier transform (DFT).
- If $\mathcal{R}_L(z)$ is approximated for low-frequency sound waves by a simple inversion, the impulse response of the feedback system of Fig. 12 will be periodic and of infinite duration. The Fourier transform of the resulting infinite length periodic impulse train will then simplify to $j\tan(kL)$.
- In comparison to traditional time-domain bore simulation techniques, the digital waveguide bore implementation offers high efficiency without sacrificing accuracy.
- The lumped open-end reflectance filter $\mathcal{R}_L(z)$ is, in general, adequately implemented using a first- or second-order digital filter. Thus, just a single short convolution/filtering operation is necessary in this model.

3.4 Cylindrical Sections: Frequency-Domain Approach

No wind instrument is constructed of a perfectly uniform cylindrical pipe. However, it is possible to approximate non-uniform bore shapes with cylindrical sections, as illustrated in Fig. 13. This approach can be pursued in either the frequency- or the time-domain.

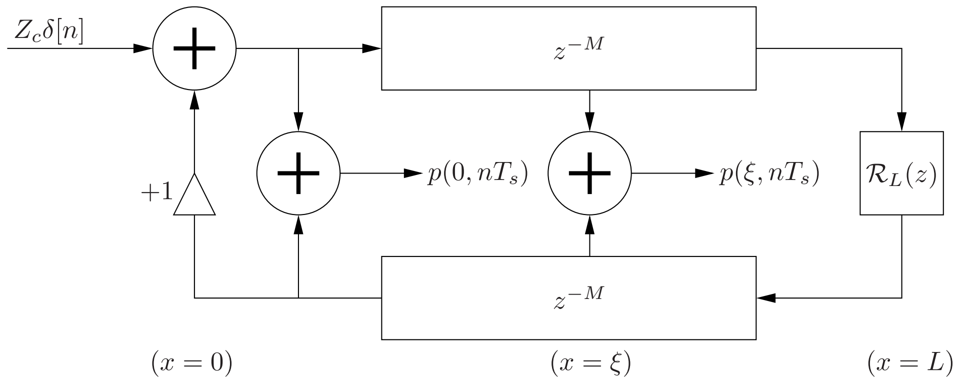
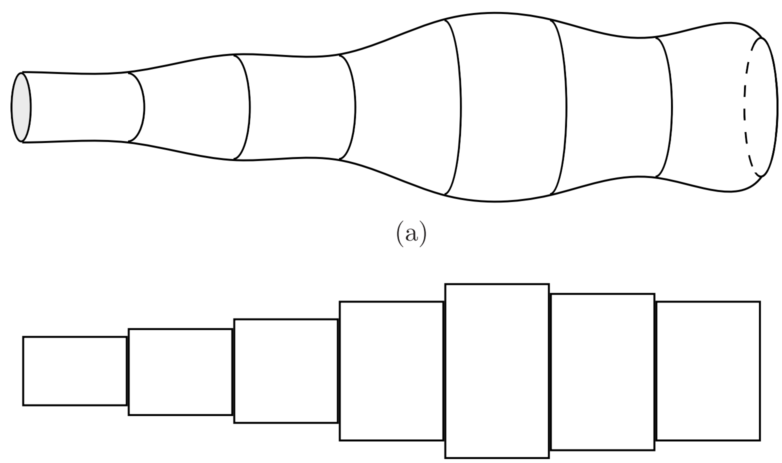


Figure 12: Digital waveguide model of a closed-open cylindrical bore.



(a)

(b)

Figure 13: A non-uniform bore (a) and its approximation in terms of cylindrical sections (b).

- A sinusoidal pressure disturbance at position x in a cylindrical pipe of finite length is given by

$$P(x,t) = \left[C^+ e^{-jkx} + C^- e^{jkx}\right] e^{j\omega t}, \tag{37}$$

  where C^+ and C^- are complex traveling-wave amplitudes, $k = \omega/c$ is the wave number, ω is radian frequency, and c is the speed of sound in air.

- The corresponding volume velocity is:

$$U(x,t) = \frac{1}{Z_c}\left[C^+ e^{-jkx} - C^- e^{jkx}\right] e^{j\omega t}, \tag{38}$$

  where $Z_c = \rho c/A$ is the real-valued wave impedance of the pipe and A is its cross-sectional area.

- For a pipe which extends from $x = 0$ to $x = L$ and is terminated at $x = L$ by the load impedance Z_L, it is possible to derive (from the equations above) an expression for the impedance at $x = 0$, or the *input impedance* of the cylindrical pipe, given by

$$Z_{in} = \frac{P(0,t)}{U(0,t)} = Z_c\left[\frac{C^+ + C^-}{C^+ - C^-}\right] \tag{39}$$

$$= Z_c\left[\frac{Z_L\cos(kL) + jZ_c\sin(kL)}{jZ_L\sin(kL) + Z_c\cos(kL)}\right]. \tag{40}$$

- This expression can also be formulated in terms of a *transfer matrix* [Keefe, 1981] which relates pressure and volume velocity at the input and output of a cylindrical section of length L as

$$\begin{bmatrix} P_0 \\ U_0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} P_L \\ U_L \end{bmatrix}, \tag{41}$$

where the lossless transfer-matrix coefficients are given by

$$\begin{aligned} a &= \cos(kL) \\ b &= jZ_c \sin(kL) \\ c &= \frac{j}{Z_c} \sin(kL) \\ d &= \cos(kL). \end{aligned}$$

- In this way, a cylindrical pipe is represented by a transfer matrix, defined by its particular length and radius.
- By comparison with Eq. (40), the input impedance of the section can be calculated from the transfer-matrix coefficients as

$$Z_{in} = \frac{b + aZ_L}{d + cZ_L}. \tag{42}$$

- For a sequence of n cylindrical sections, the input variables for each section become the output variables for the previous section. The transfer matrices can then be cascaded as

$$\begin{aligned} \begin{bmatrix} P_0 \\ U_0 \end{bmatrix} &= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdots \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} P_L \\ U_L \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_L \\ U_L \end{bmatrix} \end{aligned} \tag{43}$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \prod_{i=1}^{n} \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \tag{44}$$

and the input impedance found for the entire acoustic structure as

$$Z_{in} = \frac{B + AZ_L}{D + CZ_L}. \tag{45}$$

- Losses can be accurately accounted for in the transfer matrices by using a complex value of the wavenumber k.

3.5 Cylindrical Sections: Time-Domain Approach

- In general, abrupt diameter discontinuities rarely occur in vocal tract or wind-instrument air column profiles.
- However, these discontinuities will be encountered in the approximation of complex air column shapes using cylindrical pipe sections.
- At the boundary of two discontinuous and lossless cylindrical sections, Fig. 14, there will be a change of characteristic impedance which results in partial reflection and transfer of traveling-wave components.
- Assuming continuity of pressure and conservation of volume flow at the boundary,

$$p_1^+ + p_1^- = p_2^+ + p_2^- \tag{46}$$

and

$$\frac{1}{Z_{c1}} \left[p_1^+ - p_1^-\right] = \frac{1}{Z_{c2}} \left[p_2^+ - p_2^-\right], \tag{47}$$

where Z_{c1} is the characteristic impedance of cylindrical section 1.

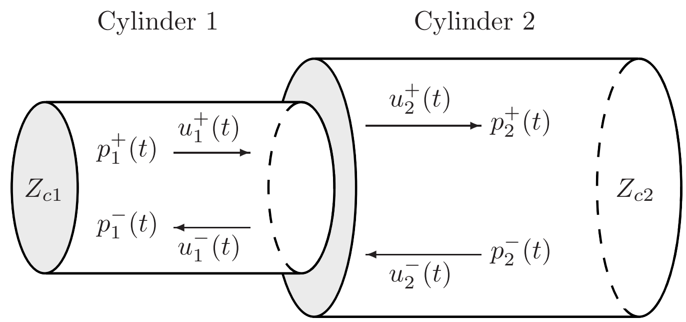


Figure 14: Junction of two cylindrical pipe sections.

- Because the characteristic wave impedance of a cylindrical pipe is real, these expressions apply to both time- and frequency-domain wave variables.
- Solving for p_1^- and p_2^+ at the junction,

$$
\begin{aligned}
p_1^- &= \left(\frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}}\right) p_1^+ + \left(\frac{2Z_{c1}}{Z_{c2} + Z_{c1}}\right) p_2^- \\
&= \mathcal{R}_{12}\, p_1^+ + (1 - \mathcal{R}_{12})\, p_2^- \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
p_2^+ &= \left(\frac{2Z_{c2}}{Z_{c2} + Z_{c1}}\right) p_1^+ - \left(\frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}}\right) p_2^- \\
&= (1 + \mathcal{R}_{12})\, p_1^+ - \mathcal{R}_{12} p_2^- \qquad (49)
\end{aligned}
$$

  where $\mathcal{R}_{12}$ is the reflectance for the junction of cylinders 1 and 2. $\mathcal{R}_{12}$ is given by

$$
\begin{aligned}
\mathcal{R}_{12} &= \frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}} \\
&= \frac{A_1 - A_2}{A_1 + A_2},
\end{aligned}
$$

  where A_1 is the cross-sectional area of section 1.
- The relationships of Eqs. (48) and (49) are referred to as *scattering equations*.
- The scattering equations are implemented by the structure shown in Fig. 15a, which was first derived for an acoustic tube model used in speech synthesis [Kelly and Lochbaum, 1962].
- Equations (48) and (49) can also be written in the form

$$
\begin{aligned}
p_1^- &= p_2^- + p_\Delta \\
p_2^+ &= p_1^+ + p_\Delta,
\end{aligned}
$$

  where

$$p_\Delta = \mathcal{R}_{12}\left[p_1^+ - p_2^-\right].$$

- In this way, the Kelly-Lochbaum scattering junction can be implemented with a single multiply, as shown in Fig. 15b [Markel and Gray, 1976].
- Smith [1987] points out that junction passivity is guaranteed for $-1 \leq r_{12}(t) = \mathcal{R}_{12}(\omega) \leq 1$.
- The digital waveguide implementation of lossless wave propagation in two discontinuous cylindrical sections is shown in Fig. 16.

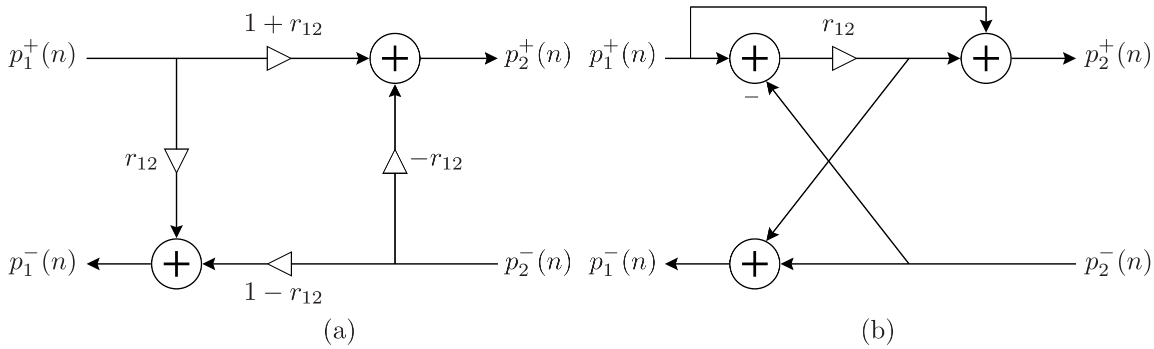


Figure 15: (a) The Kelly-Lochbaum scattering junction for diameter discontinuities in cylindrical bores; (b) The one-multiply scattering junction [after [Markel and Gray, 1976]].

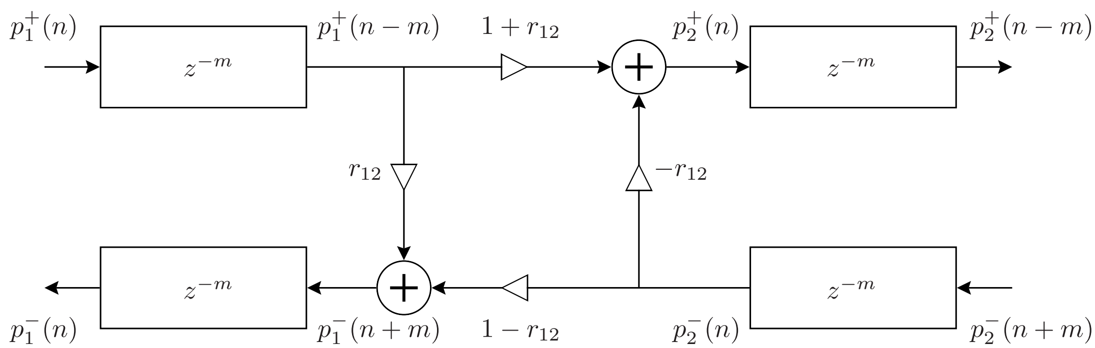


Figure 16: The digital waveguide model of two discontinuous cylindrical sections.

- In this way, any combination of co-axial cylindrical sections can be modeled using only digital delay lines and one-multiply scattering junctions.
- Thermoviscous losses can be implemented with digital filters designed in accordance with previous theoretical analyses.
- Because these models are linear and time-invariant, these loss characteristics can be commuted with an open-end filter to maximize efficiency.

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