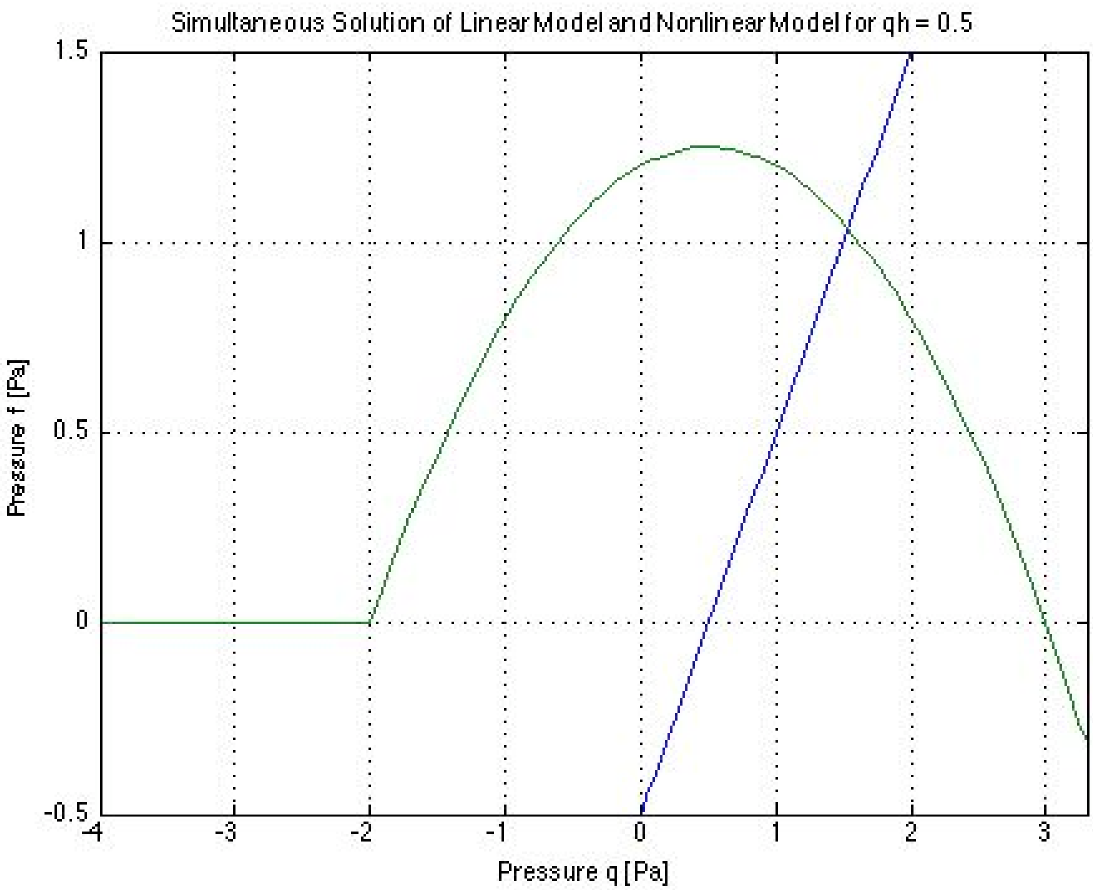
The Elements of the MSW Model

Introduction

An insightful and productive approach to the modeling of traditional acoustical instruments is based on the interaction between a passive linear portion with an active non-linear portion. The figure below shows the intersection of the linear portion (the straight line) with the non-linear portion (a truncated parabola).

The first section of this project report delves into the linear element and is followed by a brief discussion of the non-linear element; in particular, how one can



implement vibrato. The last section contains a listing of deliverables, some comments and suggestions for further work. This report includes appendices with MATLAB code and results of a literature citation search.

Defining the Elements in the Linear Model

The linear portion of the McIntyre-Shumacher-Woodhouse [MSW83] model consists of 6 interrelated elements in the time-domain and their corresponding 6 elements in the frequency-domain. The objectives of this section include:
1. naming and defining each of these 12 elements,
2. given one of these elements, showing how the other 11 follow, and
3. providing MATLAB code that computes the elements by their definition and by their analytically obtained expressions in the derivation exercise (the results are compared and appear as comments in the last section)

The complex impedance of the linear element $Z_L(\omega)$ is defined as

$$Z_L(\omega) = \frac{Q(\omega)}{F(\omega)} \quad \text{Eq. (12) MSW83,}$$

where $Q(\omega)$ and $F(\omega)$ are the Fourier transforms of $q(t)$ and $f(t)$.

$$\text{That is, } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad \text{Eq.(13) MSW83.}$$

A similar expression relates $Q(\omega)$ and $q(t)$.

For reed instruments such as the clarinet and saxophone, $q(t)$ "represents the fluctuating pressure just inside the mouthpiece" and $f(t)$ "the volume flow rate f, or volume of air flowing through the gap between reed and mouthpiece per unit time." (MSW83 p. 1327)

For bowed string instruments, such as the violin and cello, $q(t)$ "represent(s) the transverse string velocities" and $f(t)$ is "the frictional force exerted by the bow on the string." (MSW83 p. 1331)

For air-jet instruments, such as the flute and organ pipe, $q(t)$ is "the acoustic displacement of air into and out of the hole across which the jet is blowing" and $f(t)$ "is the volume flow rate in that part of the jet which is blowing into the pipe at time t." (MSW83 p. 1336)

The acoustic impulse response or Green's function $g(t)$ is the "inverse Fourier transform of $Z_L(\omega)$." (MSW83 p. 1328)

The reflection function $r(t)$ is not the same as the acoustic impulse response $g(t)$.

The acoustic impulse response is not the same as the impulse response encountered in linear systems theory where a one-input x[n] one-output y[n] system is characterized by its system transfer function $H(z)$, its frequency response $H(j\omega)$, and an impulse response h(t) which is the inverse z-transform of $H(z)$.

The reflection function "may be thought of as the disturbance that would be found at x=0 after the delta-function pulse is sent out, if the tube were terminated at x=0 by a perfect absorber such as a uniform, semi-infinite tube of the same cross section." (MSW83 p. 1327)
The acoustic impulse response is also the disturbance at x=0 but it includes the outgoing pulse as well as all the reflections of that pulse as measured at x=0. Furthermore, "$g(t)$ differs significantly from zero over a far longer time interval than $r(t)$" (MSW83 p. 1329) which has computational ramifications.

When the outgoing pressure $q_o(t)$ is equal to $\delta(t)$, the incoming pressure $q_i(t)$ is $r(t)$ by definition.
Thus,

$$q_i(t) = r(t) * q_o(t) \quad \text{Eq. (4) MSW83.}$$

Where the * denotes convolution.

At the reed, assuming superposition of plane waves, the sum of the outgoing and incoming pressures is equal to $q(t)$, the pressure just inside the mouthpiece.
That is,

$$q(t) = q_0(t) + q_i(t) \quad \text{Eq. (7) MSW83.}$$

From Newton's Second Law and assuming the superposition of pressures, it may be shown that

$$Zf(t) = q_o(t) - q_i(t). \quad \text{Eq. (8) MSW83}$$

Z is the characteristic impedance of the linear element.

Adding and subtracting Eq. (7) MSW83 and Eq. (8) MSW83 we get

$$q(t) + Zf(t) = 2q_o(t) \text{ and } q(t) - Zf(t) = 2q_i(t). \quad \text{Eq. (9) MSW83}$$

Substituting Eq. (9) MSW83 into Eq. (4) MSW83 we obtain

$$q_i(t) = r(t) * q_o(t)$$
$$\text{i.e.,} q(t) - Zf(t) = r(t) * [q(t) + Zf(t)]$$
$$\text{i.e.,} q(t) = r(t) * [q(t) + Zf(t)] + Zf(t)$$
$$\text{i.e., } q(t) = q_h(t) + Zf(t) \quad \text{Eq. (10) MSW83}$$
$$\text{where } q_h(t) = r(t) * [q(t) + Zf(t)] \quad \text{Eq. (11) MSW83}$$

Transforming Eqs. (10,11) MSW83, solving for $R(\omega)$ and using Eq.(12) MSW83 we obtain

$$Q(\omega) = R(\omega)[Q(\omega) + ZF(\omega)] + ZF(\omega)$$
$$i.e., R(\omega) = \frac{Q(\omega) - ZF(\omega)}{Q(\omega) + ZF(\omega)}$$
$$i.e., R(\omega) = \frac{Z_L(\omega) - Z}{Z_L(\omega) + Z}. \quad \text{Eq. (15) MSW83}$$

Eq. (10) MSW83 when plotted as $f(t)$ vs $q(t)$ is the equation of a straight line with slope $1/Z$ and $q(t)$-intercept given by $q_h(t)$. The intersection of this straight line with the non-linear curve provides a unique $[q(t), f(t)]$ solution. These two values are substituted into the right-hand-side of Eq. (11) MSW83 and then the convolution with $r(t)$ produces $q_h(t)$ which (with Z) defines the straight line of Eq. (10) MSW83 and thus the algorithm proceeds.

A Worked-out Example

Let $Z_L(\omega)$ be given by

$$Z_L(\omega) = Z_L(z) = \frac{1 - bz^{-1}}{1 - az^{-1}} \text{ for } z = e^{j\omega} \qquad (1)$$

From Eq.(12) MSW83 and (1),

$$Q(z) = \frac{1}{1 - az^{-1}} \qquad (2)$$

and

$$F(z) = \frac{1}{1 - bz^{-1}}. \qquad (3)$$

From Eq.(13) MSW83, (2) and (3),

$$q[n] = a^n u[n] \qquad (4)$$

and

$$f[n] = b^n u[n] \qquad (5).$$

For $Z_L(\omega)$ given as (1), the acoustic impulse response is given by

$$g[n] = a^n u[n] - ba^{n-1} u[n-1] \qquad (6).$$

From Eq. (15) MSW83 and (1), we obtain the reflection coefficient $R(z)$. That is,

$$R(z) = R(\omega) = \frac{Z_L(\omega) - Z}{Z_L(\omega) + Z} = \frac{\frac{1 - bz^{-1}}{1 - az^{-1}} - Z}{\frac{1 - bz^{-1}}{1 - az^{-1}} + Z}$$

$$= \frac{1 - bz^{-1} - Z(1 - az^{-1})}{1 - bz^{-1} + Z(1 - az^{-1})} \quad \text{for } z = e^{j\omega}. \qquad (7)$$

Taking the inverse z-transform of (7) yields an expression for the reflection function $r[n]$

$r[n] = \alpha_1 \alpha_2^n u[n] - \alpha_3 \alpha_2^{n-1} u[n-1]$ where
$\alpha_1 = (1 - Z)/(1 + Z)$, $\alpha_2 = (b + aZ)/(1 + Z)$, and $\alpha_3 = \alpha_1 (b - aZ)/(1 - Z)$. (8)

From Eq. (9) MSW83, (4) and (5), we obtain expressions for the outgoing pressure $q_o(t)$ and the incoming pressure $q_i(t)$. That is,

$$q[n] + Zf[n] = 2q_o[n]$$

$$\text{i.e., } q_o[n] = \frac{1}{2}[q[n] + Zf[n]]$$

$$\text{i.e., } q_o[n] = \frac{1}{2}\{a^n + Zb^n\}u[n] \qquad (9)$$

and

$$q[n] - Zf[n] = 2q_i[n]$$

$$\text{i.e., } q_i[n] = \frac{1}{2}[q[n] - Zf[n]]$$

$$\text{i.e., } q_i[n] = \frac{1}{2}\{a^n - Zb^n\}u[n]. \qquad (10)$$

Taking the z-transform of (9) and (10) yields

$$Q_o(z) = \frac{1}{2}\left(\frac{1}{1 - az^{-1}} + \frac{Z}{1 - bz^{-1}}\right) \qquad (11)$$

and

$$Q_i(z) = \frac{1}{2}\left(\frac{1}{1 - az^{-1}} - \frac{Z}{1 - bz^{-1}}\right). \qquad (12)$$

These 12 equations correspond to the time and frequency elements of the MSW83 linear model for the given $Z_L(z)$.

As a check on the consistency of these results, we start from (11) and (12) and then obtain (1):

From Eq. (15) MSW83, $R(z) = \dfrac{Z_L(z) - Z}{Z_L(z) + Z}$. Solving for $Z_L(z)$,

$$Z_L(z) = Z\left[\frac{1 + R(z)}{1 - R(z)}\right] = Z\left[\frac{1 + Q_i(z)/Q_o(z)}{1 - Q_i(z)/Q_o(z)}\right]$$

$$i.e., Z_L(z) = Z\left[\frac{Q_o + Q_i}{Q_o - Q_i}\right] = Z\left[\frac{\frac{1}{2}\left(\frac{1}{1 - az^{-1}} + \frac{Z}{1 - bz^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1 - az^{-1}} - \frac{Z}{1 - bz^{-1}}\right)}{\frac{1}{2}\left(\frac{1}{1 - az^{-1}} + \frac{Z}{1 - bz^{-1}}\right) - \frac{1}{2}\left(\frac{1}{1 - az^{-1}} - \frac{Z}{1 - bz^{-1}}\right)}\right]$$

$i.e., Z_L(z) = \dfrac{1 - bz^{-1}}{1 - az^{-1}}$ which was where we started. (qed)

Comparing the Defining Expressions with the Analytic Derivations of the Example

Equations (1-8) were implemented in the MATAB code which appears in the Appendix. These results were designated as the "theoretical results." Thus, there is a $g(t)$ computed from the inverse transform of $Z_L(\omega)$ and there is a $g_{th}(t)$ computed from

$$g[n] = a^n u[n] - ba^{n-1}u[n-1] \qquad (6).$$

There is coding still to be done for equations 8-12; however, there is excellent agreement between theoretical and transformed results for this example.

The Elements of the MSW Non-Linear Model

The F(q) functions in the MSW83 paper are second-order and third-order. That is,

$$F(q) = k(p-q)(q-q_c) \qquad \text{Eq. (20) MSW83}$$

$$F(q) = K(p-q)(q-q_c)(q+p-2q_c) \qquad \text{Eq. (24) MSW83}$$

The motivation for using Eq. (24) MSW83 was that the "unrealistic behavior of the model can be traced to the unrealistic symmetry of the parabola" of Eq. (20) MSW83. (MSW83, p. 1331). Rather than use the cubic model as suggested, this work utilized a non-linear $F(q)$ suggested in the paper by N. H. Fletcher [NHF 99]. In this paper, a derivation based on Bernoulli's equations, yields a cubic equation where the flow-rate $f(t)$ is linked to the pressure in the mouth p, the elastic compliance of the reed β, and the static aperture between reed and mouthpiece x_o. That is,

$$F(q) = \left[\frac{2(p-q)}{\rho}\right]^{1/2} [x_o - \beta(p-q)]W \qquad \text{Eq. (6.3) NHF99}$$

(W is the width of the aperture and ρ is the density of air.)

The derivation starts from Bernoulli's equation $p - q = \frac{1}{2}\rho v^2$ where v is the speed of air through the mouthpiece in [m/sec].
Solving for v and expressing the volume flow rate as the product of v and the cross-sectional area of the aperture, yields Eq. (6.3) NHF99.

The q_c root (the value of the air-pressure inside the mouthpiece at which the aperture closes and the volume flow-rate is zero) in Eq. (24) of MSW83 is analogous to the root of the $x_o - \beta(p-q)$ term in Eq. (6.3) NHF99. In particular,

$$q = p - \frac{x_o}{\beta} \qquad (13)$$

This suggests that a varying inside-the-mouth air pressure will vary the left-hand zero-crossing of the non-linear curve. This point also depends on the static aperture opening x_o which may be controlled by the musician's embouchure. The right-hand zero crossing of the non-linear curve depends only on q.

Vibrato is simulated in both MSWsynthREV0 and NHFsynthREV0 using the idea of varying the parameters on the right-hand-side of (13) as suggested by Gilbert, Simon and Terroir [GST05].

The dynamic variation in the solution of the linear and non-linear equations is depicted in the dynamic.m, a MATLAB script file intended to be used as a tool to study the and make refinements on the synthesis models. This code appears in Appendix F.

Solutions of the Linear and Non-Linear Model

As stated earlier the simultaneous solution of the linear model and the non-linear model is summarized as:

"Eq. (10) MSW83 when plotted as $f(t)$ vs $q(t)$ is the equation of a straight line with slope $1/Z$ and $q(t)$-intercept given by $q_h(t)$. The intersection of this straight line with the non-linear curve provides a unique $[q(t), f(t)]$ solution. These two values are substituted into the right-hand-side of Eq. (11) MSW83 and then the convolution with $r(t)$ produces $q_h(t)$ which (with Z) defines the straight line of Eq. (10) MSW83 and thus the algorithm proceeds."

In this work, we use a closed-form solution of Eq. (20 MSW83 and the straight-line of Eq. (10) MSW83 using the well-known quadratic formula. The MATLAB code from MSWsynthREV0.m (which appears in Appendix E) is as follows

```
Eta = (1-k*(p-qh)*Z+k*Z*(qh-qc));
Theta = k*(p-qh)*(qh-qc);
Zeta = k*Z^2;
f = (-Eta+sqrt(Eta^2+4*Zeta*Theta))/(2*Zeta);
```

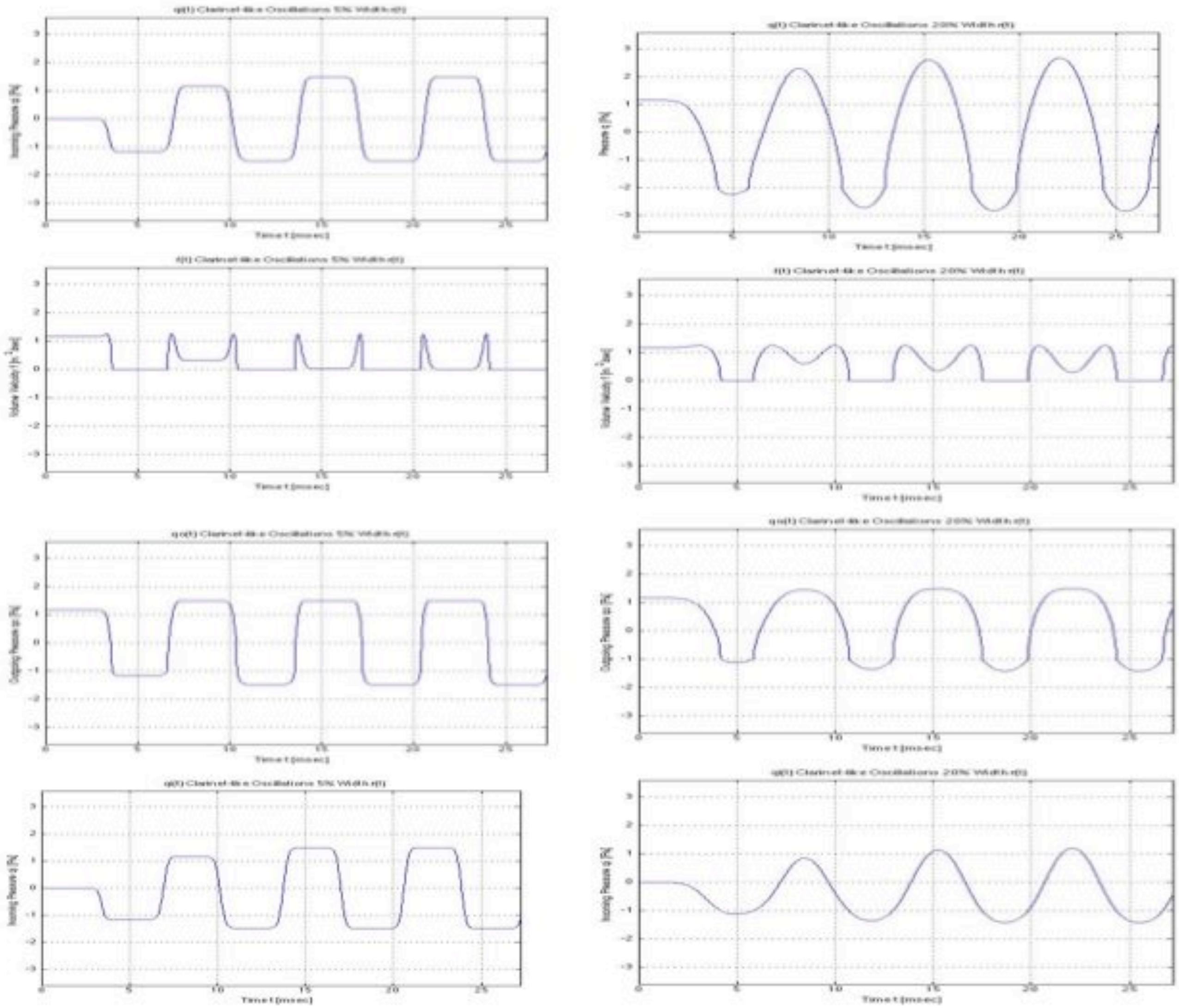
The closed-form solution of a cubic is not so well known but readily obtainable. This solution appears in the NHFsynthREV0.m code with follows in Appendix D. The cubic solution portion is repeated here.

```
            A = (a*p*beta^2 - 2*a*xo*beta + 2*a*p*beta^2 - 1)/(-
a*beta^2);
            B = (2*a*p*xo*beta - 2*a*p^2*beta^2 - a*xo^2 +
2*a*p*xo*beta - a*beta^2*p^2 + 2*qh)/(-
     a*beta^2);
            C = (a*p*xo^2 - 2*a*p^2*xo*beta + a*p^3*beta^2 -
qh^2)/(-a*beta^2);

            P = B - A^2/3.;
            Q = C + (2*A^3-9*A*B)/27.;
            U = (-Q/2. - sqrt(Q^2/4 + P^3/27.))^(1/3);

            q = real((U - P/3./U - A/3.));
```

Generating The MSW83 Figure 5 Plots



These plots agree favorably with those presented in MSW83 and were obtained using a constant inside-the-mouth air pressure of 3, k = 0.2, the 5% and 20% Gaussian reflection functions and the parabolic nonlinear function.

The following table summarizes some of the model elements that appear in the literature.

	$F(q)$		$r(t)$		A
clarinet	$k(p-q)(q-q_c)$	(20) MSW83	$a\exp[-b(t-T)^2]u(t)$	(18) MSW83	-1
clarinet	$K(p-q)(q-q_c)(q+p-2q_c)$	(24) MSW83	$a\exp[-b(t-T)^2]u(t)$	(18) MSW83	-1
clarinet	$\left[\frac{2(p-q)}{\rho}\right]^{\frac{1}{2}}[x_o-\beta(p-q)]W$	(6.3) NHF99	N/A	N/A	N/A

$A=\int_0^\infty r(t)dt$ Eq. (6) MSW83

Comments

The **deliverables** include:

- A script file “tool” called “dynamic” that currently runs on NHFsynthREV0 results to show how the solution point and the nonlinear curve vary with time.
- Sound files output from MSWsynthREV0 and NHFsynthREV0 which include GST05 vibrato.
- A worked analytical example which links the parameters of the linear model.

Further work is alluded to and tagged by a *.

The choice of $Z_L(z)$ in the example is constrained so that the Fourier transform exists because the time-frequency relationships of the elements of the linear model are expressed as forward and inverse Fourier transform pairs. This constraint forces the roots of both the numerator and denominator polynomials of $Z_L(z)$ to be within the unit circle due to the zeros of $Z_L(z)$ being the poles of $F(z)$. In signal processing vernacular, we would state that $Z_L(z)$ be minimum phase; i.e., both poles and zeros located within the unit circle.
*The implications of this observation may be significant in the choice of model parameters in a synthesis scenario.

In the worked-out example, (1), (2) and (3) were given by

$$Z_L(\omega)=Z_L(z)=\frac{1-bz^{-1}}{1-az^{-1}} \text{ for } z=e^{j\omega} \qquad (1)$$

$$Q(z) = \frac{1}{1 - az^{-1}} \qquad (2)$$

and

$$\mathrm{F(z)} = \frac{1}{1 - bz^{-1}}. \qquad (3)$$

For the given (1), equations (2) and (3) could just have well been written as

$$Q(z) = 1 - bz^{-1} \qquad (14)$$

and

$$\mathrm{F(z)} = 1 - az^{-1} \qquad (15)$$

In which case, $q[n] = \delta[n] - b\delta[n-1]$ and $f[n] = \delta[n] - a\delta[n-1]$.

This would lead to different expressions for the (9) and (10):

$$q_o[n] = \frac{1}{2}[q[n] + Zf[n]]$$

i.e., $q_o[n] = \frac{1}{2}\{\delta[n](1+Z) - \delta[n-1](b + aZ)$ (16)

and

$$q_i[n] = \frac{1}{2}[q[n] - Zf[n]]$$

i.e., $q_i[n] = \frac{1}{2}\{\delta[n](1-Z) - \delta[n-1](b - aZ)$. (17)

Then, taking the z-transform, would yield different expressions for (11) and (12):

$$Q_o(z) = \frac{1}{2}\left(1 + Z - (b + aZ)z^{-1}\right) \qquad (18)$$

and

$$Q_i(z) = \frac{1}{2}\left(1 - Z - (b - aZ)z^{-1}\right). \qquad (19)$$

*It remains to be shown that as before, a check on the consistency of these results would occur, when we start from (18) and (19) and then obtain (1).

An additional condition on the reflection function $r(t)$ is that

$$\int_0^\infty r(t')dt' = -1. \qquad \text{Eq. (6) MSW83}$$

This is a statement of "the fact that according to linear acoustic theory there can be no permanent, steady difference in pressure between the interior of the tube and the air outside." (MSW83 p. 1328) The reflection function was computed in three different ways in the simulation. In all three cases, the sum of the reflection function samples equaled 0.6 An extension of this work would be to analytically compute the infinite sum of (8), to evaluate the expression comparing to 0.6, and then adjusting the parameters, a, b, and Z so that the *sum equals negative one. No attempt was made in the example to represent an actual physical system. Rather, the exercise was intended to be mathematically tractable while presenting the various concepts in the model and demonstrating how one could calculate any particular element in the model from knowledge of the complex impedance $Z_L(\omega)$ and the characteristic impedance Z.

*At various places in the MSW83 paper there are references to more extensive work. An extension of this paper would be to read, evaluate and implement some of the results in those papers.

Since the MSW83 paper (26 years) there have been numerous extensions of their work (notably the physical modeling with Digital Waveguides (DWG) of the Stanford University group – the research results of Julius Orion Smith and Gary Scavone [GPS97] in *particular). A natural extension of this work would be to use DWG models of tone-holes, non-cylindrical bores, dispersion, etc. and then to obtain the reflection function when the outgoing wave is an impulse, and then using this waveform for the r(t) in the MSW83 code. Specifically, it would be nice to generate the complex impedance as we did in the homework using the time-domain approach with scattering junctions for a comparable physical system (same length and bore) and then use the resulting reflection function in place of the Gaussian in the MSW code. This would involve digging into the reflpoly(z) function that generates the reflection coefficient magnitude and length correction for an un-flanged cylindrical bore as given by Levine and Schwinger [LS48].

A casual survey of available papers on the MSW83 yielded 136 citations. A listing (with some abstracts) appears in Appendix C.

*An extension of this work would be to compare the cubic MSW83 with the cubic NHF99 both algebraically and in terms of how well they simulate the reed instrument.

*Some effort was made to produce a nice-sounding vibrato by varying the depth of the vibrato over time. An extension of this would be to vary the vibrato rate over time; starting slowly to approach some nominal tasteful vibrato.

*Some effort was also made to produce controllable attack and release transients. This remains an area of interest and probably will benefit from the use of the "dynamic.m" script tool or something like it.

With the view of processing the digitized output of a microphone placed in the far-field of a playing acoustic instrument, it is useful to model where the output is present in the model. One thought is that the reflection function filter (as implemented as an M-order

*FIR) is divided in two such that the output taken at the first half represents the pressure available at the open end of the clarinet. To access this point in the FIR filter, it seems that the FIR of M-order should be reconfigured as the cascade of two M/2-order FIR. The following MATLAB code suggests how this could be accomplished.

```
%filterSplit.m
%
% splits an M-order FIR into two cascadable M/2-order FIR
filters
%
% the idea to access the filter at the middle for output
purposes
%
% 9 DEC 2009
b_test = [1 2 3 4 5 6 7 6 5 4 3 2 1];
r_test = roots(b_test);

M = length(b_test) - 1;
b1 = poly(r_test(1:M/2));
b2 = poly(r_test(M/2+1:M));

impulse = [1 zeros(1,2*M)];

h = filter(b_test,1,impulse);
h_test = filter(b1,1,filter(b2,1,impulse));
```

*The Table which summarized non-linear and linear model equations could be extended as the literature continues to be read.

*Hysterisis could accommodate two paths through the non-linear curve based on the physics of the history. Something like a Schmidtt trigger which in the onset transient portion could account for the initial slapping of the reed against the mouthpiece as it closes for the first time where the musician has used an impulsive burst of air pressure to start the oscillation. For some non-linear curves, the hysterisis extension is clearer than with other (for example, the tanh x function resembles the Schmidtt trigger curve).

Bibliography

[MSW83] 1. M. E. McIntyre, R. T. Schumacher, and J. Woodhouse, *On the oscillations of musical instruments*, J. Acoust. Soc. Am. 74(5) November 1983, p. 1325-1345.

[GPS97] Gary P. Scavone, *An Acoustic Analysis of Single-Reed Woodwind Instruments with an Emphasis on Design and Performance Issues and Digital Waveguide Modeling Techniques*, PhD thesis, Music Dept., Stanford University, March, 1997.

[LS48] Levine, H. and Schwinger, J. (1948). *On the radiation of sound from an unflanged circular pipe*. Phys. Rev., 73(4):383–406.

[NHF99] N. H. Fletcher, *The nonlinear physics of musical instruments*, Rep. Prog. Phys, 63, (1999) 723-764.

[GST05] J. Gilbert, L. Simon, and J. Terroir, *Vibrato of saxophones*, J. Acoust. Soc. Am. 118 (4), October 2005.

Appendix A. linearModel.m : MATLAB code for the example of the linear model.

Appendix B. MSWfigFIVErev2.m : MATLAB code for generating figure 5 of the MSW83 paper

Appendix C. Citations (136) of the MSW83 paper in the literature.

Appendix D. NHFsynthREV0.m : Synthesis using the NHF non-linear synthesis model

Appendix E. `MSWsynthREV0.m :` MATLAB code for synthesizing using the MSW parabola for the non-linear element

Appendix F. dynamic.m : plotting program of the solution point as the model is dynamically synthesizing an output