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Computational Modeling of Musical Acoustic Systems
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Isospectral Drum Investigation

Wave equations and physical-modeling algorithms are typically found together. When referring to a Riemannian manifold being excited with a rigid boundary, or edge of a drum, a wave propagates from the point of origin and bounces from boundary to boundary until all the concentrated wave energy in the system diffuses and dissipates.

For any point on the surface of a Riemannian manifold, we can relate its location to the manifold in terms of an x coordinate and y coordinate. Points inside and on the boundary are separated, because the rigid boundary has no movement and it has a constant rate of diffusion which acts on the surface of the drum. We evaluate the movements of the points inside the boundary while the membrane moves and stretches over time as the wave travels towards the boundary and back using the finite difference method. If we take a snapshot of this concentration of energy at two points, the diffusion theory can be represented as follows:

$$\frac{\partial P_\Omega}{\partial t} = \frac{1}{2}\ \nabla^2 P_\Omega,$$

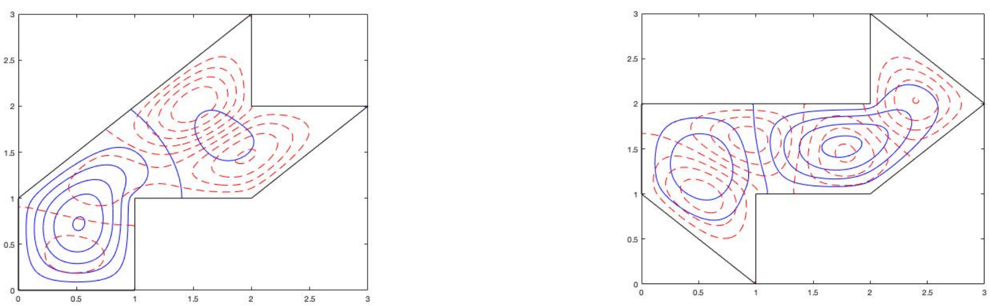
Mark Kac, 1966

The available frequencies that make up the sound of our system are the eigenvalues of the Laplacian. Furthermore, the concentration of energy can also be expressed in the form of eigenvalues and normalized eigenfunctions as the sum of the eigenfunction at the point of interest as in the following:

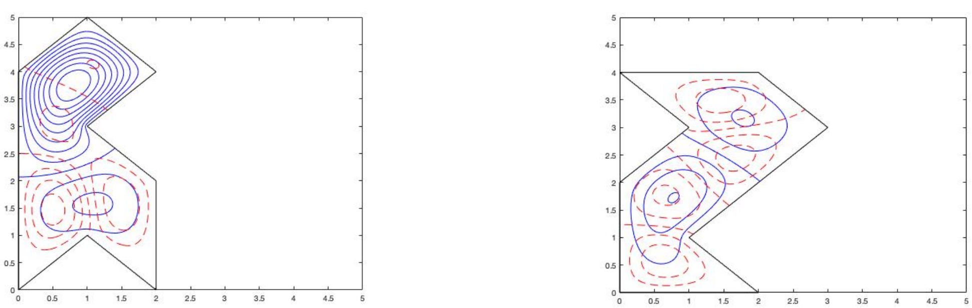
$$P_\Omega(\vec{\rho}\,|\,\vec{r};\, t) = \textstyle\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(\vec{\rho})\psi_n(\vec{r})$$

Mark Kac, 1966

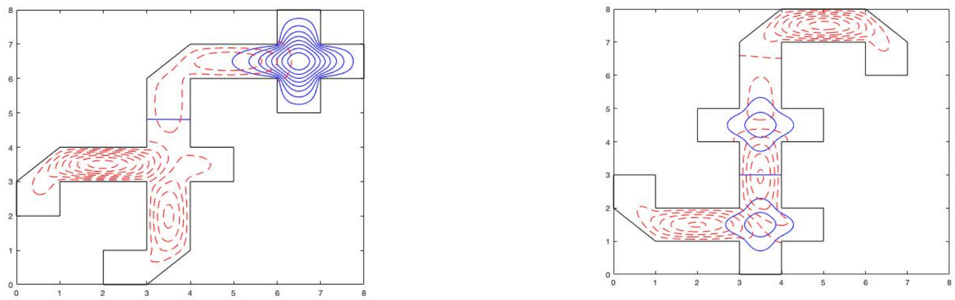
In 1966, Mark Kac asked if it was possible to have two drum shapes that shared the same set of eigenvalues and, if they did, would they sound the same or would we hear a difference between them. Years later in Gordon et al. (1992), the authors discovered Riemannian manifolds, or drum shape pairs, that shared the same Dirichlet spectra and Neumann spectra. This effectively means that the first isospectral drums in two dimensions were found. Using multiple functions within Matlab, we can evaluate those shapes and observe if they share the same eigenvalues. We can also plot their modes of vibration and compare them to one another.



Figures 1 and 2 – First pair of drum shapes with overlapping modes 2 (blue solid lines) and 4 (red dashed lines)



Figures 3 and 4 – Same as figures 1 and 2, but the drum shapes are flipped and rotated with overlapping modes 2 (blue solid lines) and 4 (red dashed lines).



Figures 5 and 6 – Isospectral figures in the Gordon et al. (1992) paper with overlaying modes 2 (blue solid lines) and 4 (red dashed lines)

Let us inspect the lines that start and end at the boundary because those lines delineate places where the membrane would not move for a specific mode of vibration. If we strike the drum at any point on those lines, we are cancelling that particular mode of vibration by exciting that section of the shape which would normally remain stationary for that mode. If two or more modes are cancelled by striking such intersections in one shape and an equivalent intersection at the same modes does not exist in the other shape, there should be an audibly noticeable difference, even more so in the lowest modes of vibration. We can observe, when overlaying two vibrating modes, that some of these lines overlap in figure 1 but not in their isospectral counterpart in figure 2, and that the same occurs in figure 3 but not in figure 4. As a result, we can determine that areas exist on the surface of these drums that have no equivalent sounding counterpart in its isospectral pair and, as we stack modes on top of each other, we uncover more of these as well. Ultimately, it should be possible to differentiate the sound between two isospectral drums because of the difficulty in finding equivalent points to strike on both surfaces.

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