EE 5345
Biomedical Instrumentation
Lecture 20: slides 376-395

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slides can be viewed at:
http:// www.seas.smu.edu/~cd/ee5345.html

Examples (cont.)

Suppose two events $A \subset \Omega, B \subset \Omega$ are not mutually exclusive:

$$A \cap B \neq f$$

Then

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

proof:

mutually exclusive

$$A \cup B = A \cup \overline{A} \cap B \qquad B = A \cap B \cup \overline{A} \cap B$$

$$\Pr(A \cup \overline{A}B) = \Pr(A) + \Pr(\overline{A} \cap B) \qquad \Pr(B) = \Pr(A \cap B) + \Pr(\overline{A} \cap B)$$

$$\Rightarrow \Pr(A \cup B) = \Pr(A) + \Pr(A) - \Pr(A \cap B) \quad \square$$

Examples (cont.)

if $A \subset \Omega$ then $\overline{A}$ is the event corresponding to "A did not occur", and

$$\Pr(\overline{A}) = 1 - \Pr(A)$$

ex) 1 roll of a fair die

if A = {roll is even} then $\overline{A}$ = {roll is odd}

Pr(A) = 1 - Pr($\overline{A}$) = 0.5

Examples (cont.)

ex) A fair coin is tossed 3 times in succession.

Events: *A*- get a total of 2 heads
B- get a head on second toss

Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
A:		x	x		x			
B:	x	x			x	x		

$\Pr(A) = 3/8 \quad \Pr(B) = 4/8 \quad \Pr(A \cap B) = 2/8$

$$\Pr\left(A \cup B\right) = 3/8 + 4/8 - 2/8 = 5/8$$

Conditional Probability

$$\Pr(A|B) \equiv \frac{\Pr(A \cap B)}{\Pr(B)}$$

ex) A fair coin is tossed 3 times in succession.

Events: *A*- get a total of 2 heads
B- get a head on second toss

Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
A:		x	x		x			
B:	x	x			x	x		

Pr(*B*) = 4/8, Pr($A \cap B$) = 2/8, Pr(A | B) = (2/8) / (4/8) = 1/2

Examples (cont.)

ex) A fair die is thrown once:

$\Omega = \{1, 2, 3, 4, 5, 6\}$

- A- roll a “2”
- B- roll is even
- $\Pr(A) = 1/6$ $\Pr(B) = 3/6$ $\Pr(A \cap B) = \Pr(A) = 1/6$

$$P(A \mid B) = (1/6)/(3/6) = 1/3$$

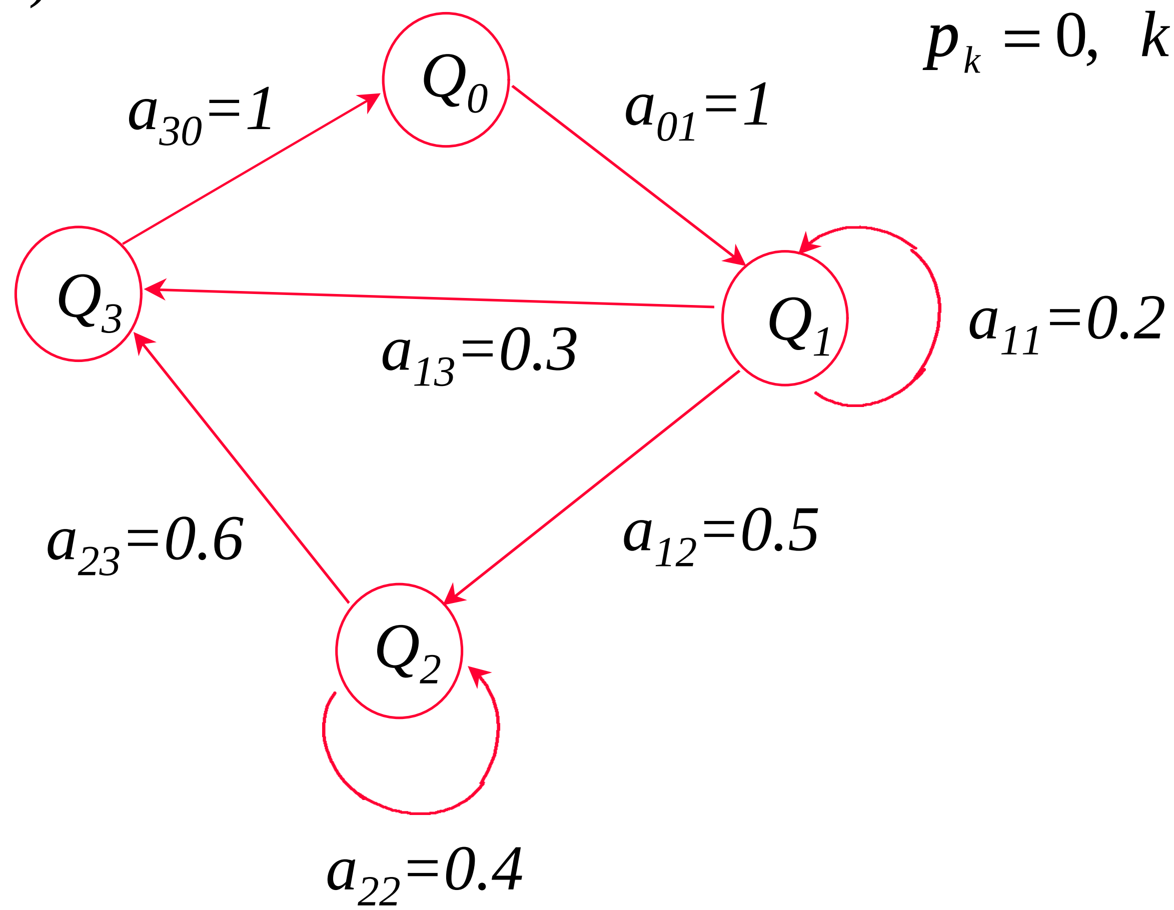
note $\Pr(A \mid A) = 1$, and if A and B are independent events:

$$\Pr(A|B) = \Pr(A)$$

Hidden Markov Models (HMM's)

example 1)

$$p_0 = 1$$
$$p_k = 0, \quad k \neq 0$$



Example of an HMM

- The a_{ij} are *state transition probabilities*, give the probability of moving from state i to state j.
- Note that:

$$\sum_j a_{ij} = 1$$

- At state Q_i, one of 3 output symbols, R, B, or Y is generated with probabilities $b_i(R), b_i(B),$ or $b_i(Y)$

State, Q_i	$b_i(R)$	$b_i(B)$	$b_i(Y)$
0	0.3	0.2	0.5
1	0.7	0.2	0.1
2	0.9	0	0.1
3	0.2	0.8	0

Example of an HMM (cont.)

- One output symbol is generated per state (like a Moore state machine).

possible output sequence: R, Y, B, B, R, Y, R, ...
state: Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2, ...

- Often the observed output symbols bear no obvious relationship to the state sequence (*i.e.* states are “hidden”).
- Knowing the state sequence generally provides more useful information about the characteristics of the signal being analyzed than the observed output symbols (as was the case with syntactic recognition).

Definition of Hidden Markov Models

- there are T observation times: $t = 0, \ldots, T\text{-}1$
- there are N states: $Q_0, \ldots, Q_{N-1}$
- there are M observation symbols: $v_0, \ldots, v_{M-1}$
- state transition probabilities:

$$a_{ij} = \Pr\left(Q_j \text{ at time } t+1 \mid Q_i \text{ at time } t\right)$$

- symbol probabilities:

$$b_j(k) = \Pr\left(v_k \text{ at time } t \mid Q_j \text{ at time } t\right)$$

- initial state probabilities:

$$p_i = \Pr\left(Q_i \text{ at } t = 0\right)$$

Definition of Hidden Markov Models (cont.)

- Define the matrices *A*, *B*, and Π:

$$\{A\}_{ij} = a_{ij}, \quad i, j = 0, \ldots, N-1$$

$$\{B\}_{jk} = b_j(k), \quad j = 0, \ldots, N-1, \quad k = 0, \ldots, M-1$$

$$\{\Pi\}_i = p_i, \quad i = 0, \ldots, N-1$$

notation for HMM: $l = (A, B, \Pi)$

- Notation for observation sequence: $O = O_0, O_1, \ldots, O_{T-1}$
- Notation for state sequence: $I = i_0, i_1, \ldots, i_{T-1}$

Three Fundamental Problems

- Problem 1: Given the observation sequence $O = O_0, O_1, \ldots, O_{T-1}$ and the model l = (A, B, Π), how do we compute the probability of the observation sequence, $Pr(O \mid \lambda)$?
- Problem 2: Given the observation sequence $O = O_0, O_1, \ldots, O_{T-1}$ and the model l = (A, B, Π), how do we estimate the state sequence, $I = i_0, i_1, \ldots, i_{T-1}$ which produced the observations?
- Problem 3: How do we adjust the model parameters l = (A, B, Π) to maximize $Pr(O \mid \lambda)$?

Relevance to Normal/Abnormal ECG Rhythm Detection

- Suppose we have one HMM that models normal rhythm, and a second HMM that models abnormal rhythm, and we have a measured observation sequence. Problem 1 can be used to determine which is the most likely model for the measured observations, hence, we can classify the rhythm as normal or abnormal.
- Suppose we have a single model which enables us to associate certain states with with the components of the ECG (P, QRS, and T waves). Problem 2 can be used to estimate the states from the observation sequence. The state sequence can then be used to detect P, QRS, and T waves.

Relevance to Normal/Abnormal ECG Rhythm Detection (cont.)

- Problem 3 is used to generate the model parameters that best fit a given training set of observations. In effect, the solution to Problem 3 allows us to build the model. This problem must be solved first before we can solve Problems 1 and 2. Problem 3 is more difficult to solve than Problems 1 and 2.

Markovian Property of State Sequences

- The sequence $i_0, i_1, \ldots, i_{T-1}$ has the Markov property:

$$\Pr\left(i_k \mid i_{k-1}, i_{k-2}, \ldots, i_0\right) = \Pr\left(i_k \mid i_{k-1}\right)$$

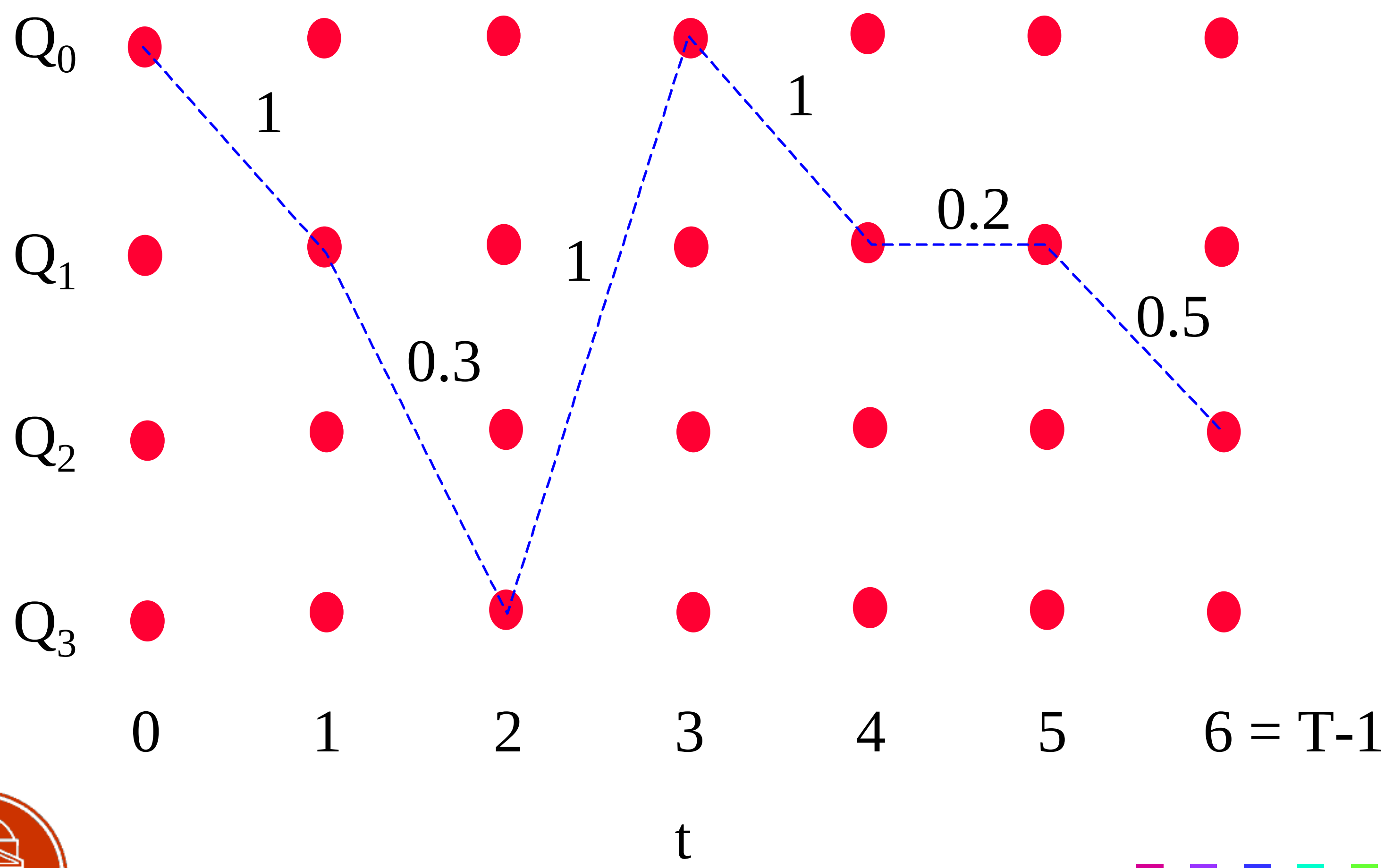
that is, the state at time $t = k$, i_k, is independent of all previous states except i_{k-1}.

- A consequence of this property is (homework):

$$\Pr\left(i_k, i_{k-1}, i_{k-2}, \ldots, i_0\right) = \Pr\left(i_k \mid i_{k-1}\right)\Pr\left(i_{k-1} \mid i_{k-2}\right)\cdots\Pr\left(i_1 \mid i_0\right)\Pr\left(i_0\right)$$

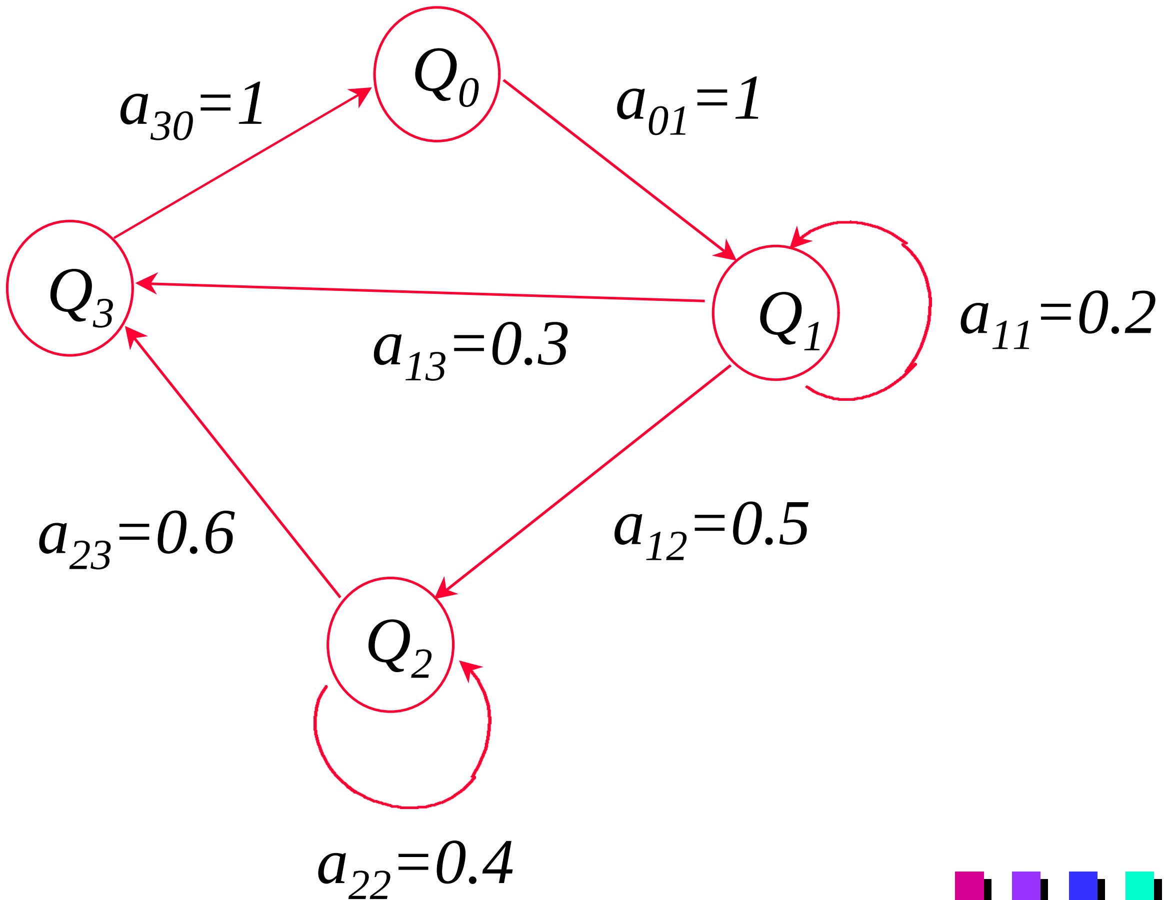
notation: $\Pr\left(i_k, i_{k-1}, i_{k-2}, \ldots, i_0\right) \equiv \Pr\left(i_k \cap i_{k-1} \cap i_{k-2} \cap \ldots \cap i_0\right)$

Trellis Representation of HMM in Example 1



Probability of state sequence: $I = Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2$

$\Pr(Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2) = 1*0.3*1*1*0.2*0.5 = 0.03$



Probability of a given *I* and *O*: $\Pr(I \cap O)$

observed output sequence: $R, \quad Y, \quad B, \quad B, \quad R, \quad Y, \quad R$

state: $Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2$

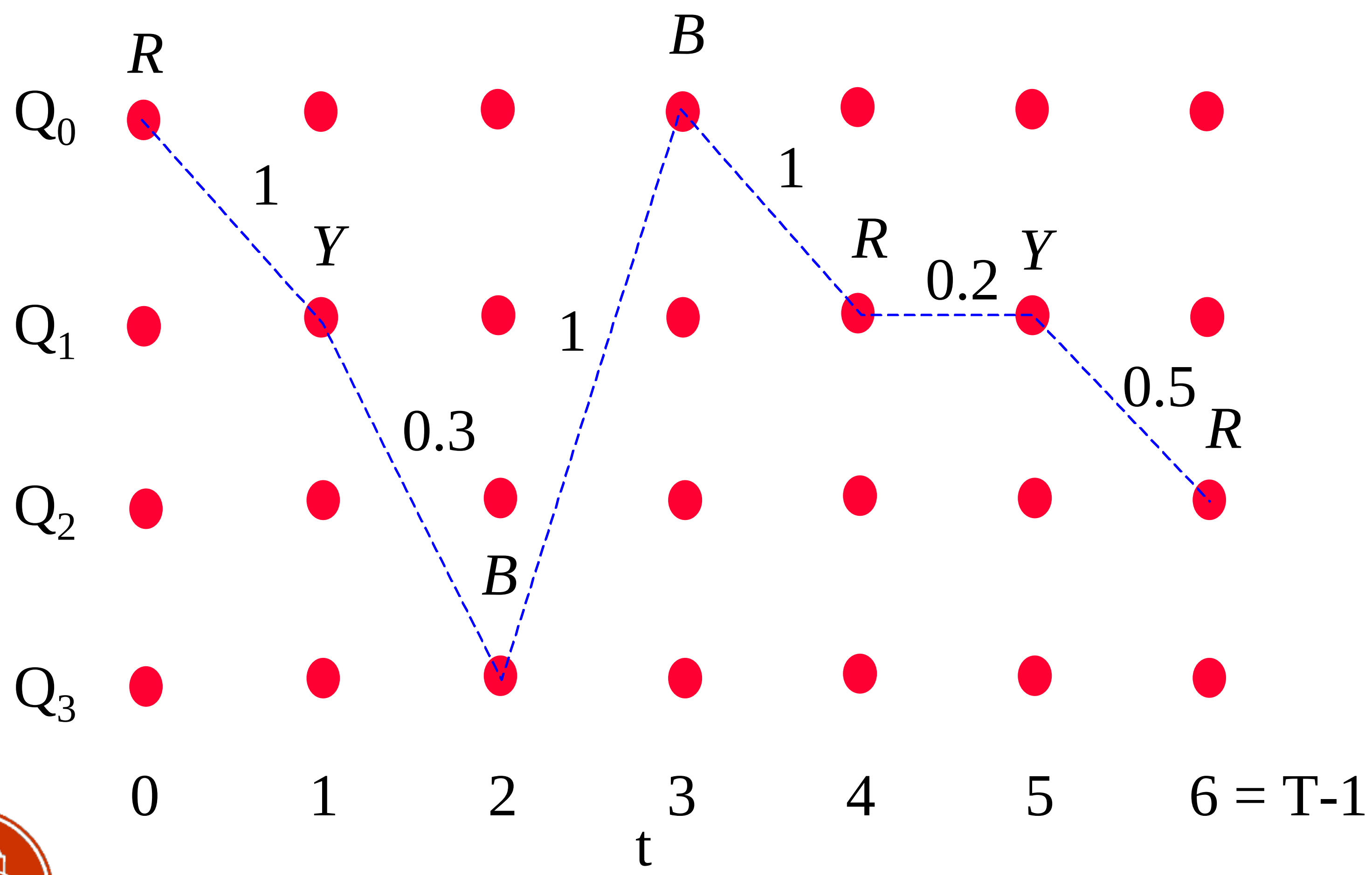
Note that:

$$\Pr(I \cap O) = \Pr(I)\Pr(O|I)$$

Back to Example 1

output sequence: $R, \; Y, \; B, \; B, \; R, \; Y, \; R$

state: $Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2$



Example (cont.)

output sequence: $R, \quad Y, \quad B, \quad B, \quad R, \; Y, \; R$

state: $Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2$

$$\Pr(I \cap O) = \Pr(I)\Pr(O|I)$$

$\Pr(I) = \Pr(Q_0, Q_1, Q_3, Q_0, Q_1, Q_1, Q_2)$

$= 1*0.3*1*1*0.2*0.5 = 0.03$

$\Pr(O \mid I) = \Pr(R, Y, B, B, R, Y, R)$

$= 0.3*0.1*0.8*0.2*0.7*0.1*0.9 = 0.0003024$

State, Q_i	$b_i(R)$	$b_i(B)$	$b_i(Y)$
0	0.3	0.2	0.5
1	0.7	0.2	0.1
2	0.9	0	0.1
3	0.2	0.8	0