The number of distinct planar configurations of *n* circles, if circles can be outside or inside one another, is:

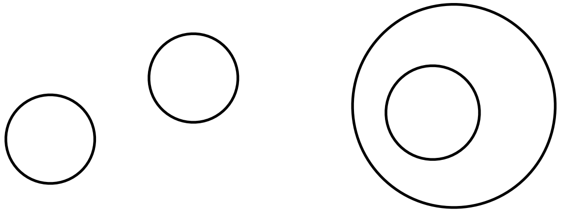
1,2,4,9,20,48...

(A00081 in Sloane's encyclopedia,
"number of rooted trees with *n* nodes")

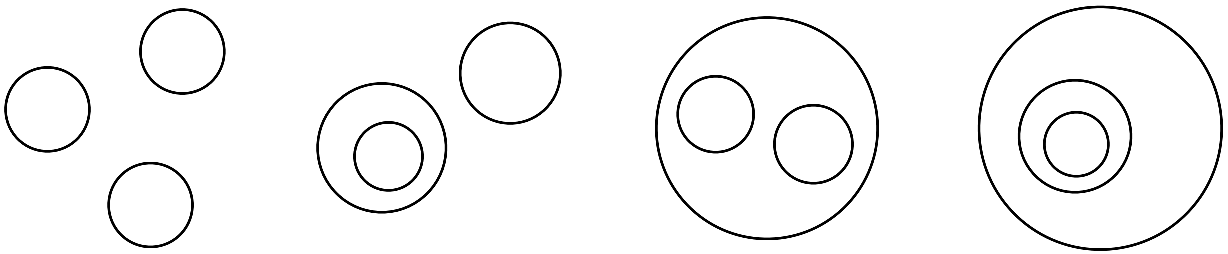
Formulas and recurrence relations are given there.

Here are illustrations of the first few terms.

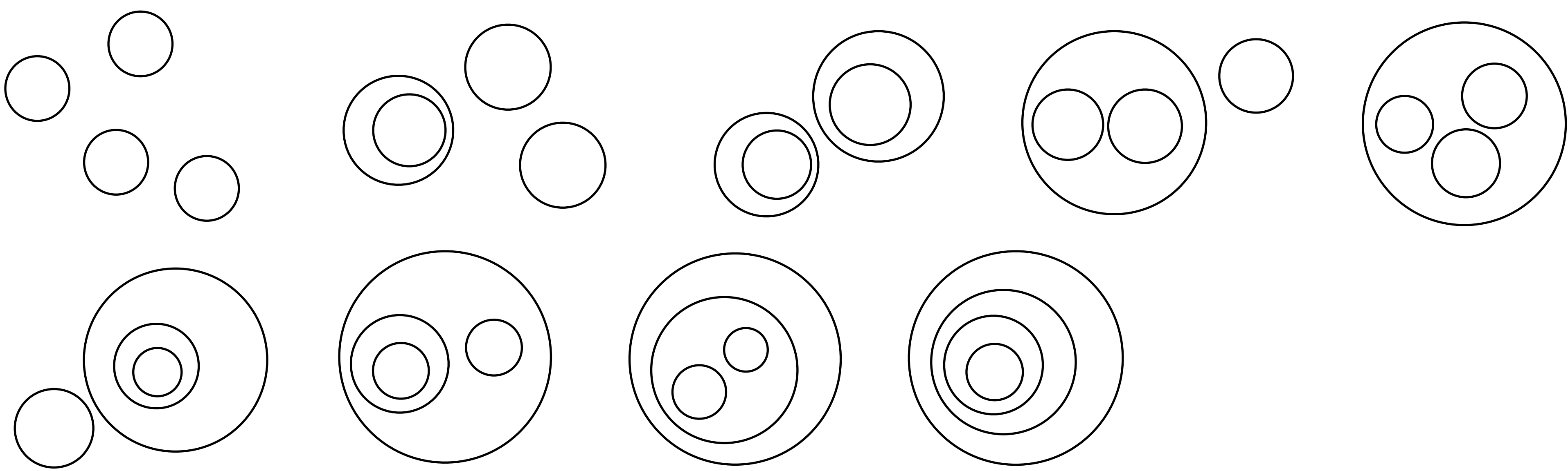
n=2:
2



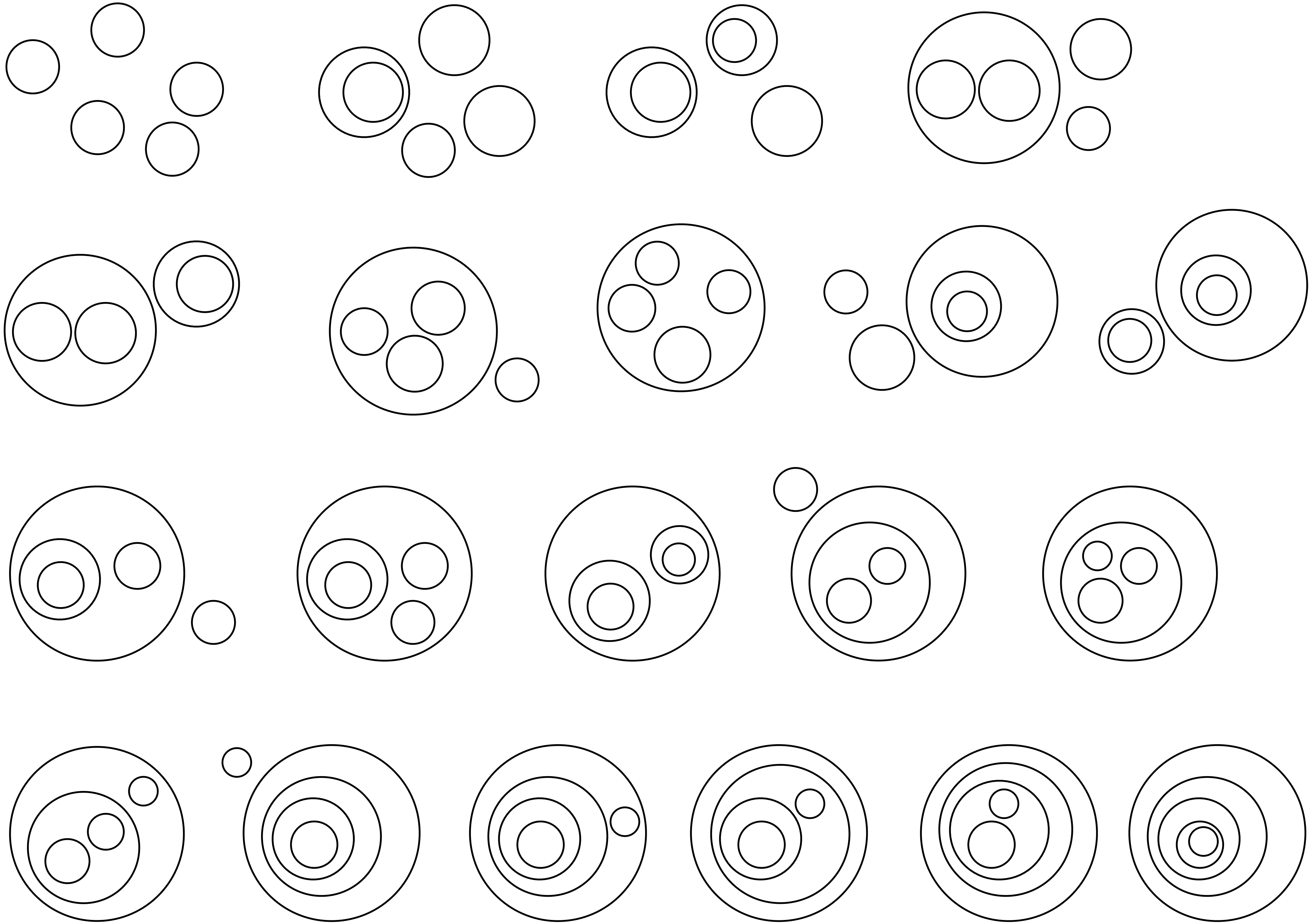
n=3:
4



n=4:
9

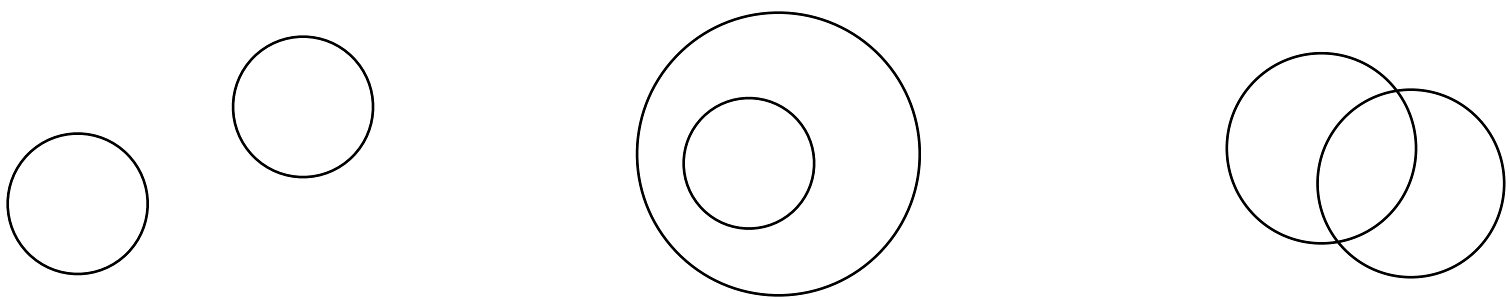


n=5:
20



What if we allow a third relationship between circles, in which they intersect?

Now $f(2) = 3$:



2-1 2-2 2-3

(The labels beneath the configurations identify them as the 1st, 2nd and 3rd configuration of 2 circles, respectively.)

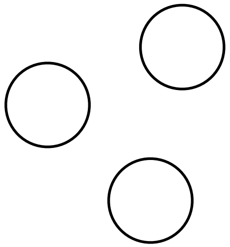
$f(3) = 14$.

On the next slide I give a label to each of the 14 3-configurations, and show their 2-decks (i.e. the list of labels for the 2-configurations that remain when you take away one circle from the configuration).

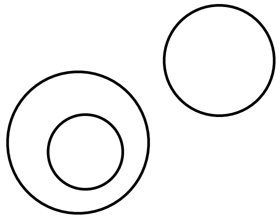
Some formalities first:

No two circles may meet in only one point (i.e. no tangent circles);
no three circles may meet in the same point.

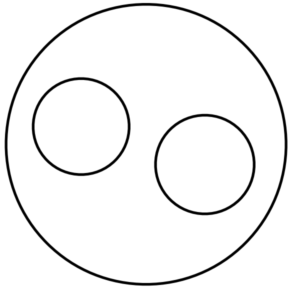
Two configurations are equivalent if there is a smooth transformation that starts with the first configuration, and ends at either the second configuration or the mirror image of the second configuration, where the smooth transformation consists of a combination of scaling and translation of each circle such that no points of intersection are created or destroyed at any intermediate moment of the transformation, and all points of intersection remain distinct throughout the smooth transformation.



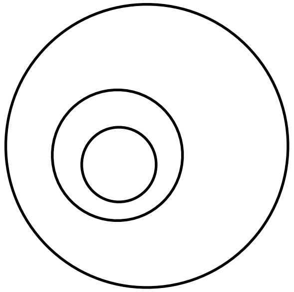
3-1
1,1,1



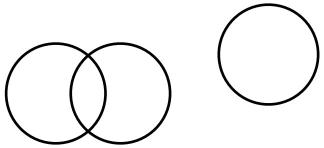
3-2
1,1,2



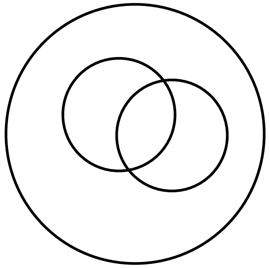
3-3
1,2,2



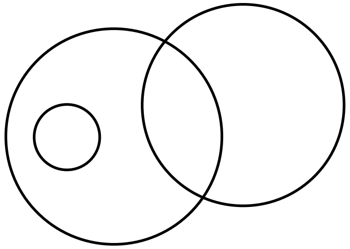
3-4
2,2,2



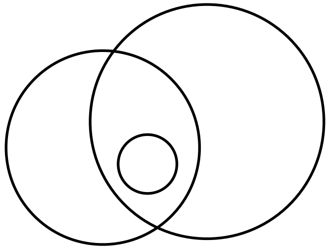
3-5
1,1,3



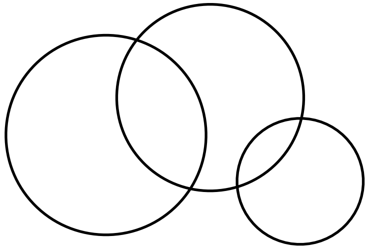
3-6
2,2,3



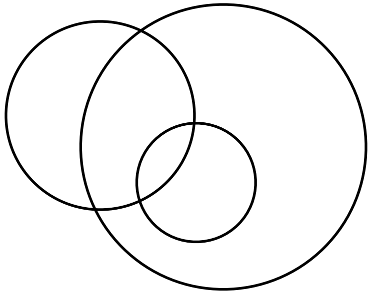
3-7
1,2,3



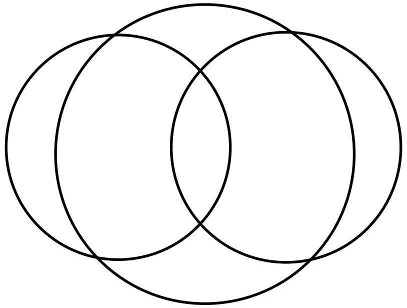
3-8
2,2,3



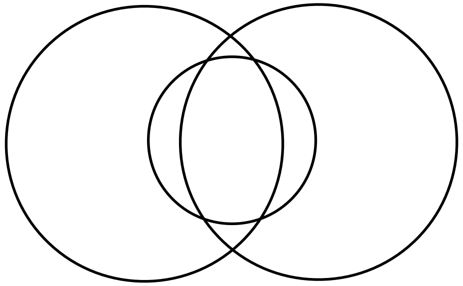
3-9
1,3,3



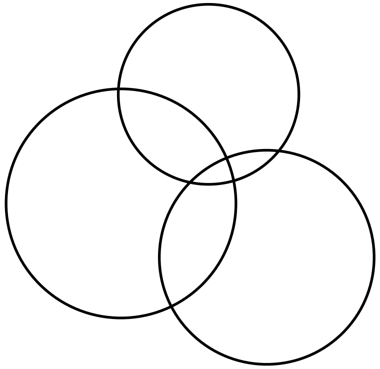
3-10
2,3,3



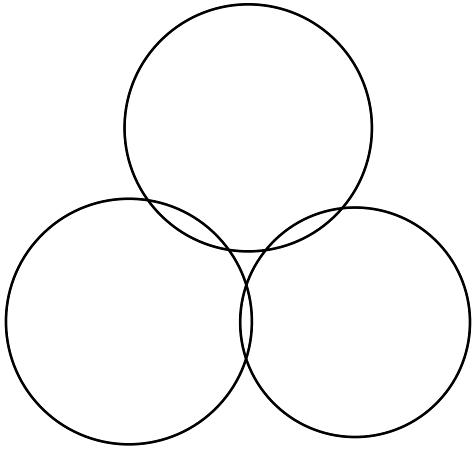
3-11
3,3,3



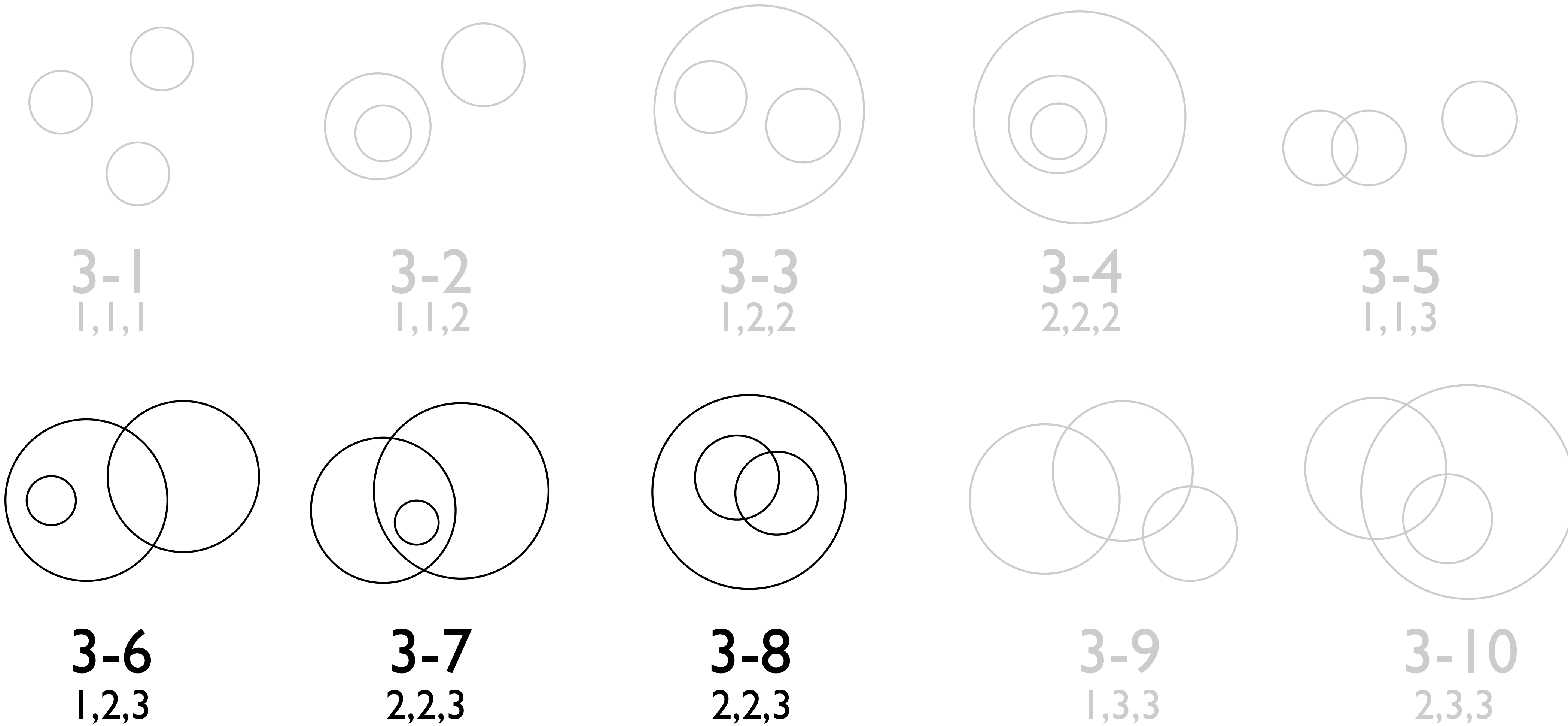
3-12
3,3,3



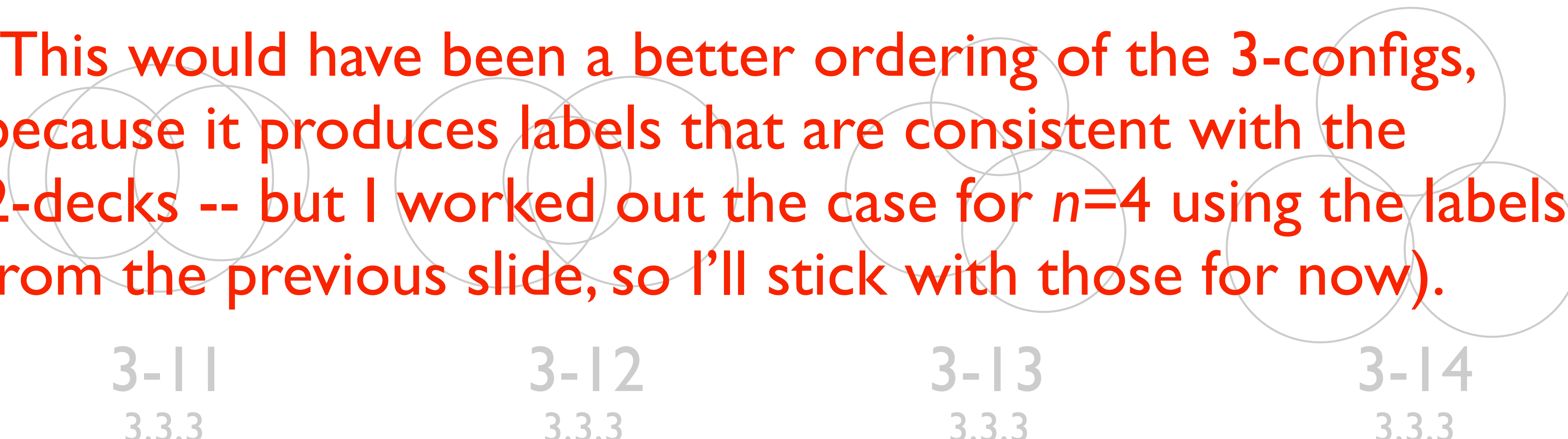
3-13
3,3,3



3-14
3,3,3

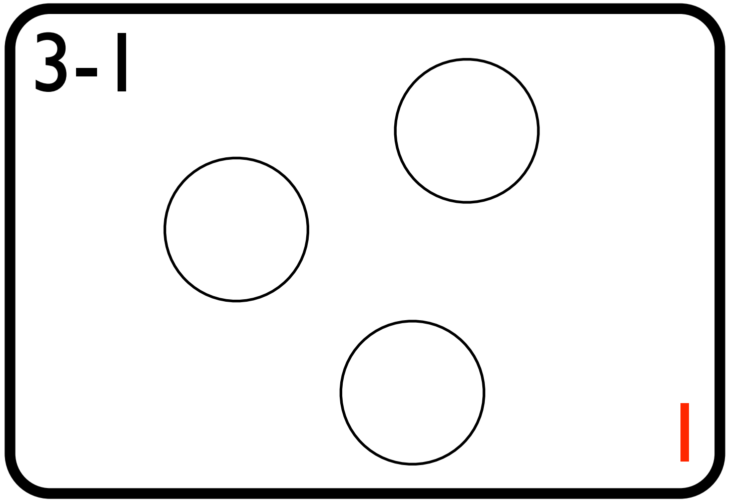


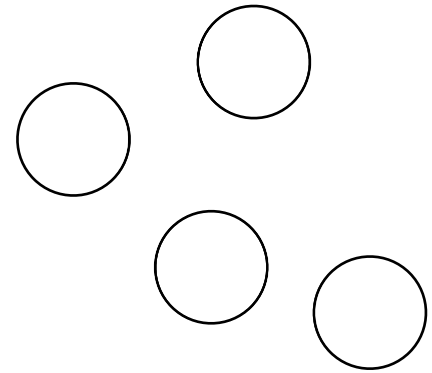
(This would have been a better ordering of the 3-configs, because it produces labels that are consistent with the 2-decks -- but I worked out the case for *n*=4 using the labels from the previous slide, so I'll stick with those for now).



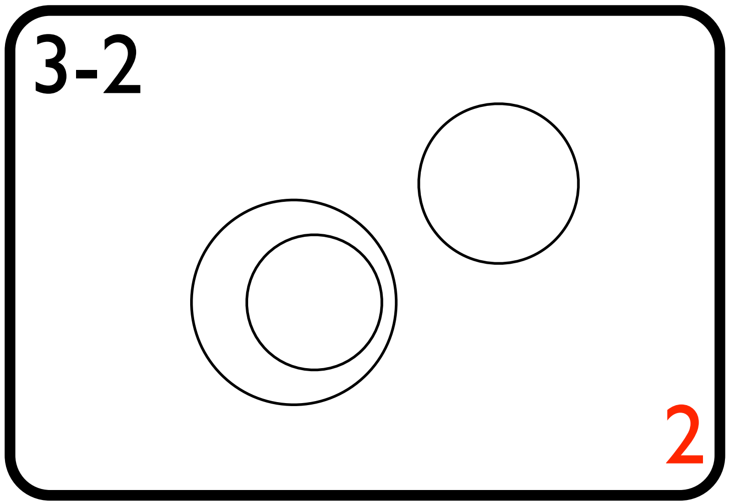
In what follows I take each of the 3-configurations and find all of the 4-configurations generated from them such that their 3-decks do not contain any 3-configurations not yet examined. That way I will never generate the same 4-configuration from two different 3-configurations. I give the 3-decks using the labels from above, and I order the generated configurations according to their 3-decks.

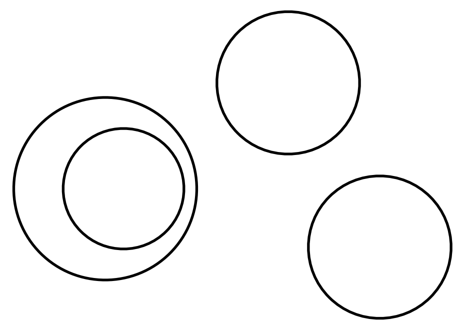
The red numbers inside the squares give the total number of 4-configs generated from each 3-config. Ignore the pink numbers and the blue numbers for now. A green asterisk means the configuration has left- and right-handed versions (not counted as distinct for now).



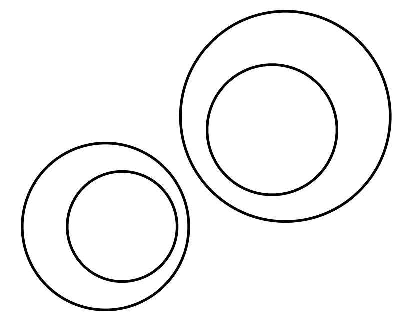


1,1,1,1

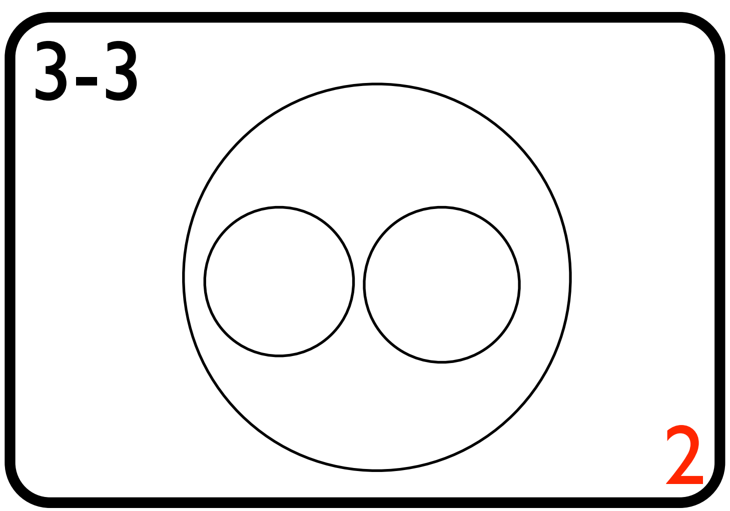


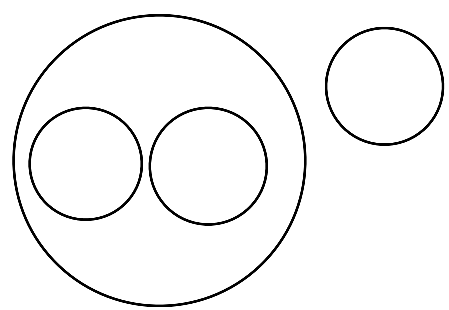


1,1,2,2

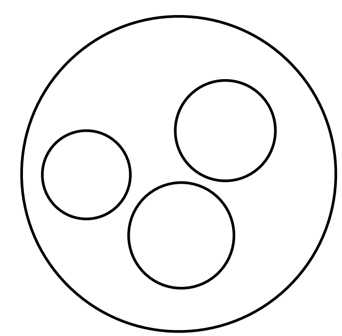


2,2,2,2



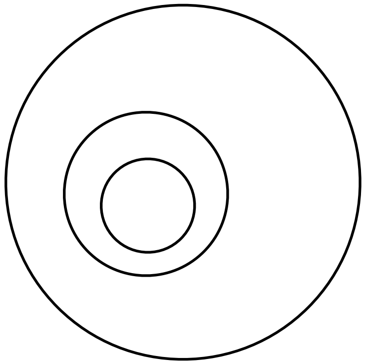


1,2,2,3

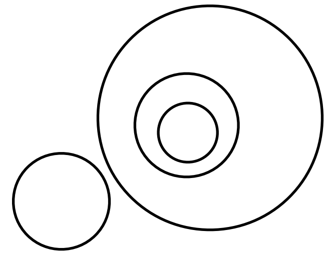


1,3,3,3

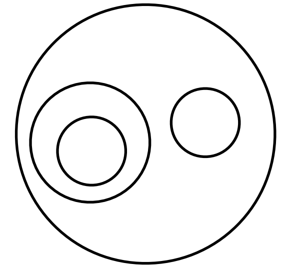
3-4



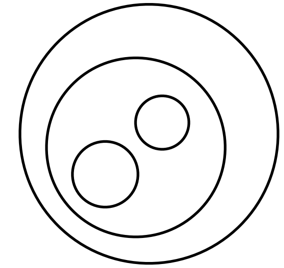
4



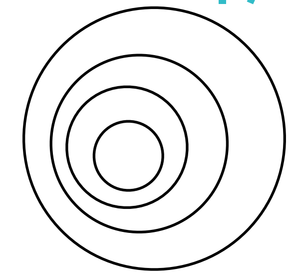
2,2,2,4



2,3,3,4

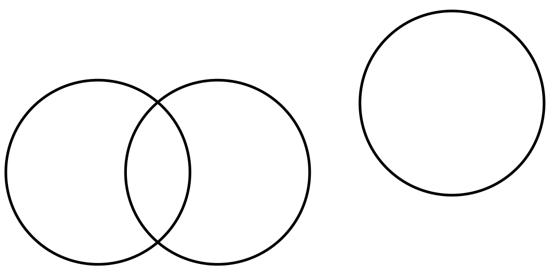


3,3,4,4

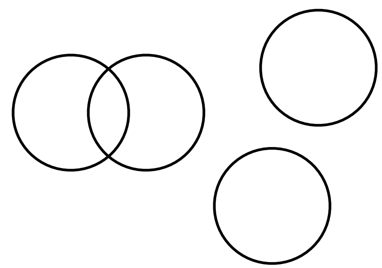


4,4,4,4

3-5

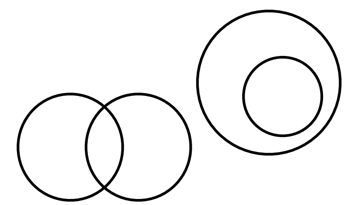


3



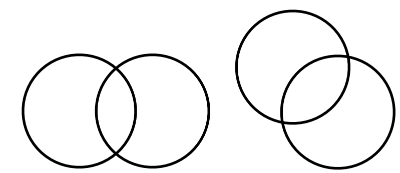
1,1,5,5

2 2



2,2,5,5

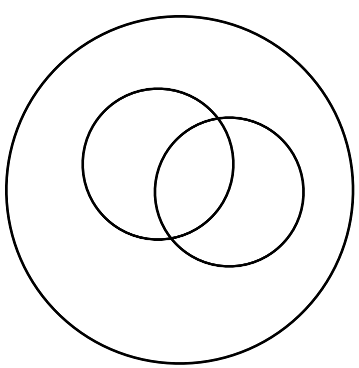
2 2



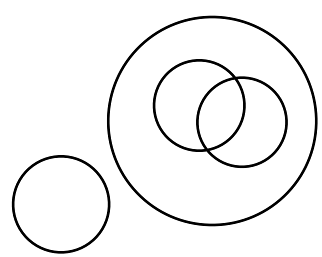
5,5,5,5

3 5

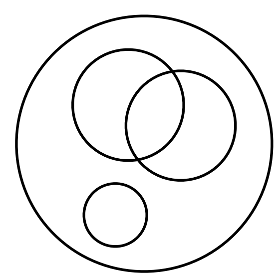
3-6



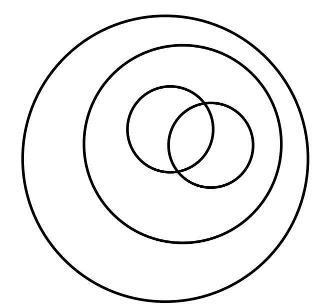
3



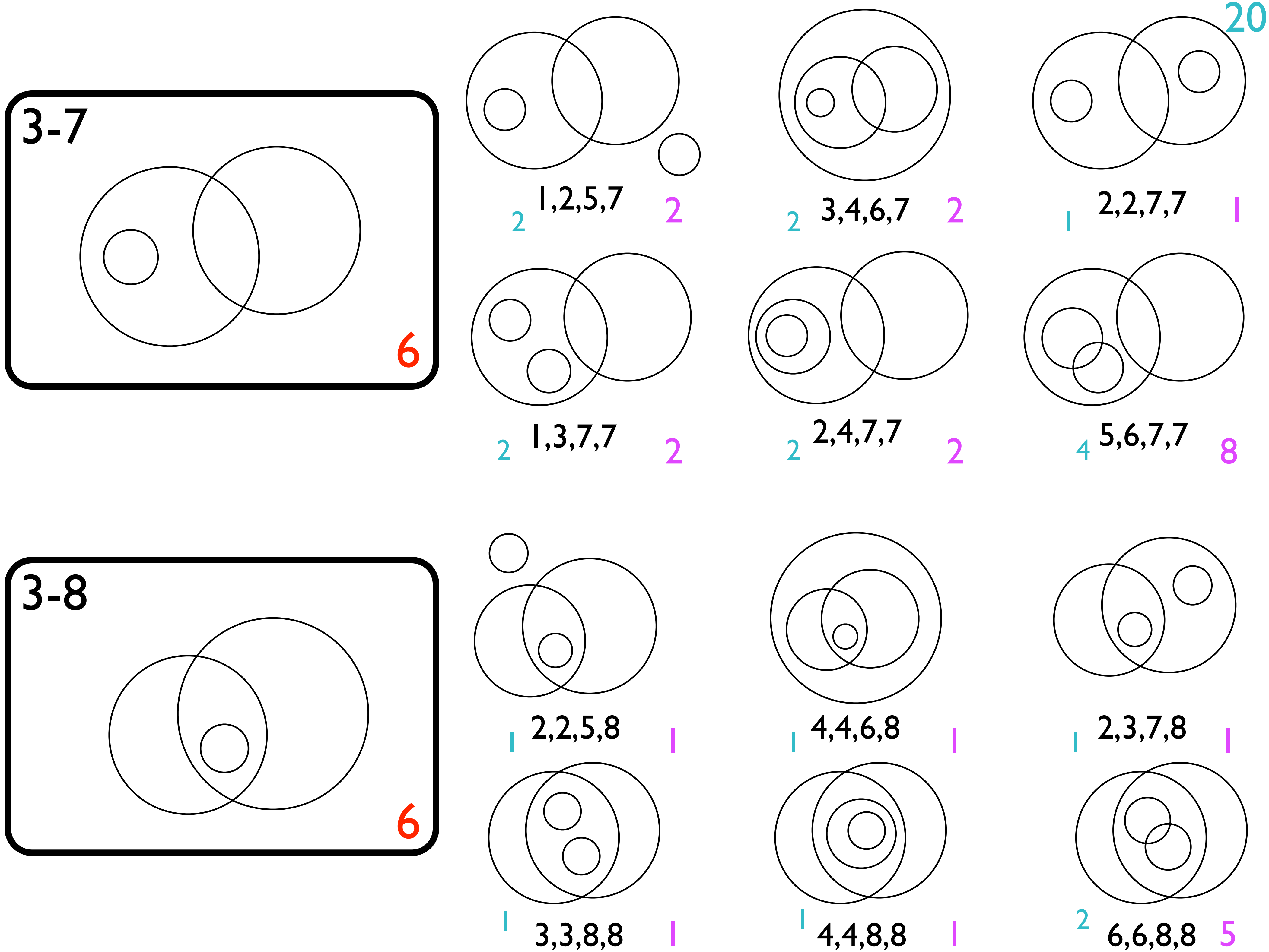
2 2,2,5,6 2



2 3,3,5,6 2

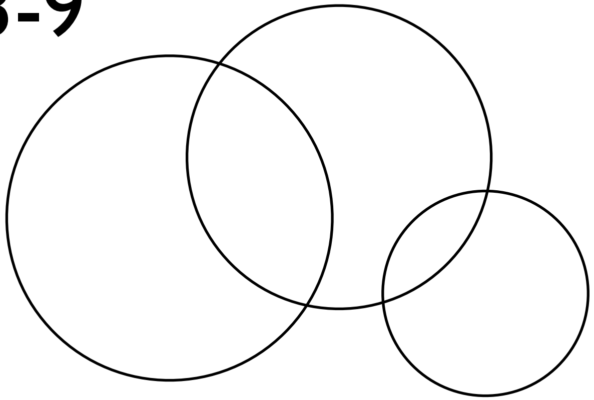


2 4,4,6,6 2

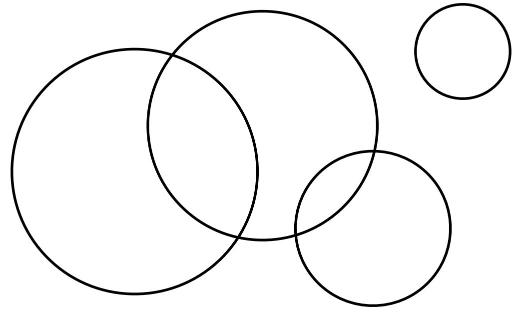

20
3-7
6
2 1,2,5,7 2
2 3,4,6,7 2
1 2,2,7,7 1
2 1,3,7,7 2
2 2,4,7,7 2
4 5,6,7,7 8
3-8
6
1 2,2,5,8 1
1 4,4,6,8 1
1 2,3,7,8 1
1 3,3,8,8 1
1 4,4,8,8 1
2 6,6,8,8 5

21

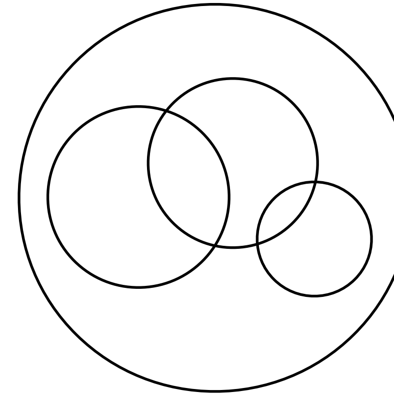
3-9



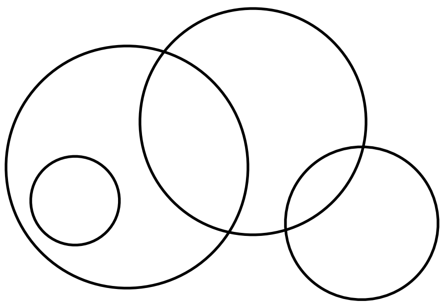
8



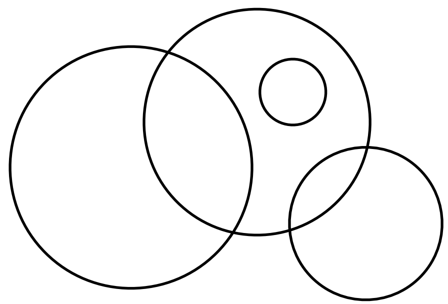
3 1,5,5,9 4



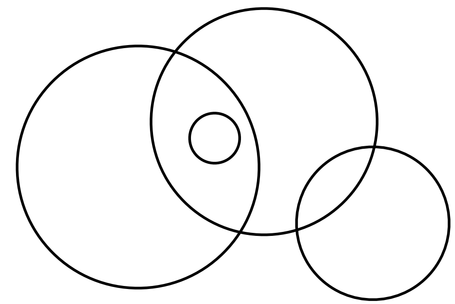
3 3,6,6,9 4



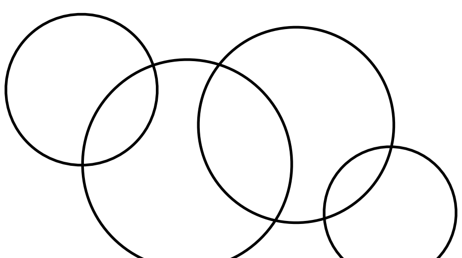
2 2,5,7,9 5



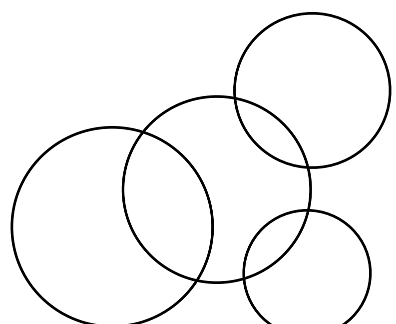
3 1,7,7,9 5



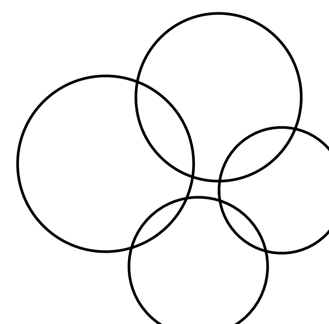
2 2,7,8,9 5



3 5,5,9,9 11

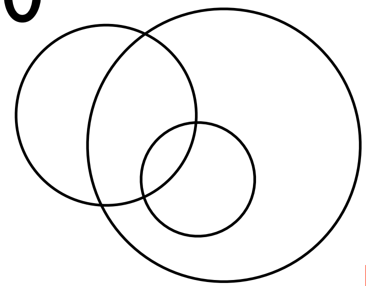


4 1,9,9,9 9 1*



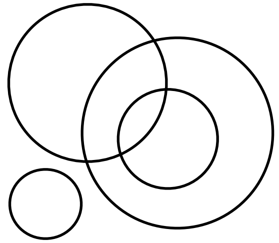
1 9,9,9,9 5

3-10

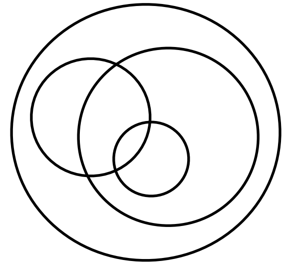


16

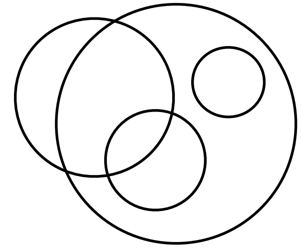
22
1*



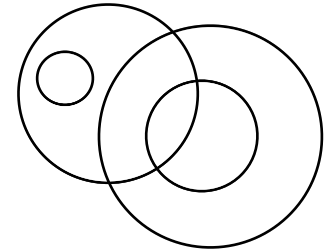
2 2,5,5,10 5



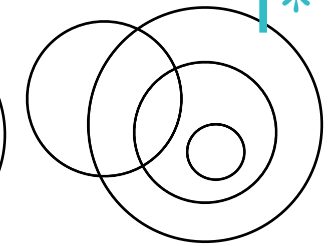
2 4,6,6,10 5



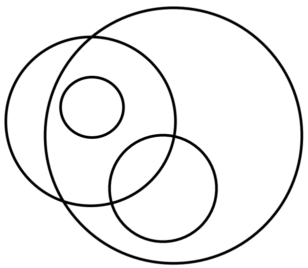
2 3,5,7,10 5



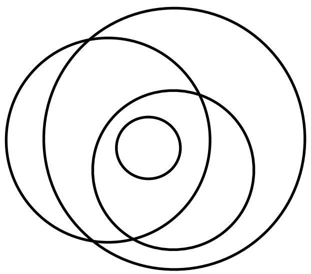
2,7,7,10 2



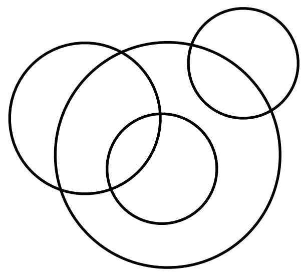
1 4,7,7,10 3



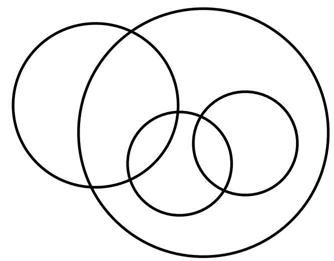
2 3,7,8,10 5



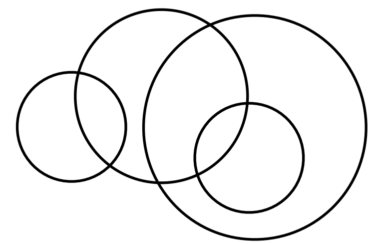
1 4,8,8,10 3



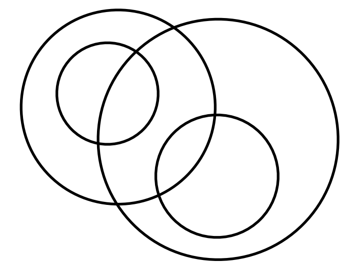
4 5,7,9,10 17



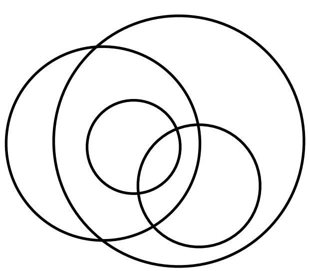
2 6,7,9,10 11



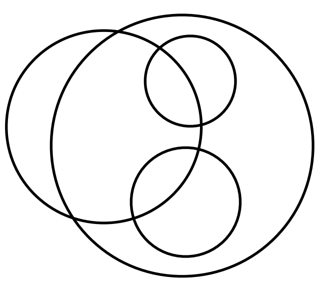
2,9,9,10 8



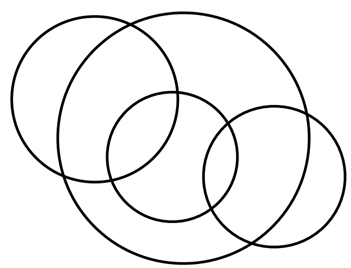
7,7,10,10 2



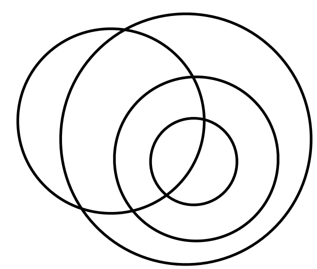
2 6,8,10,10 11



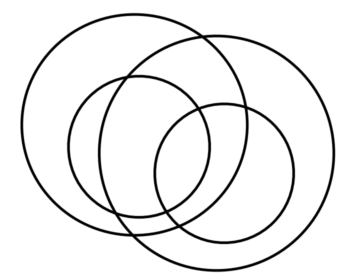
3 3,9,10,10 11
1* 6*



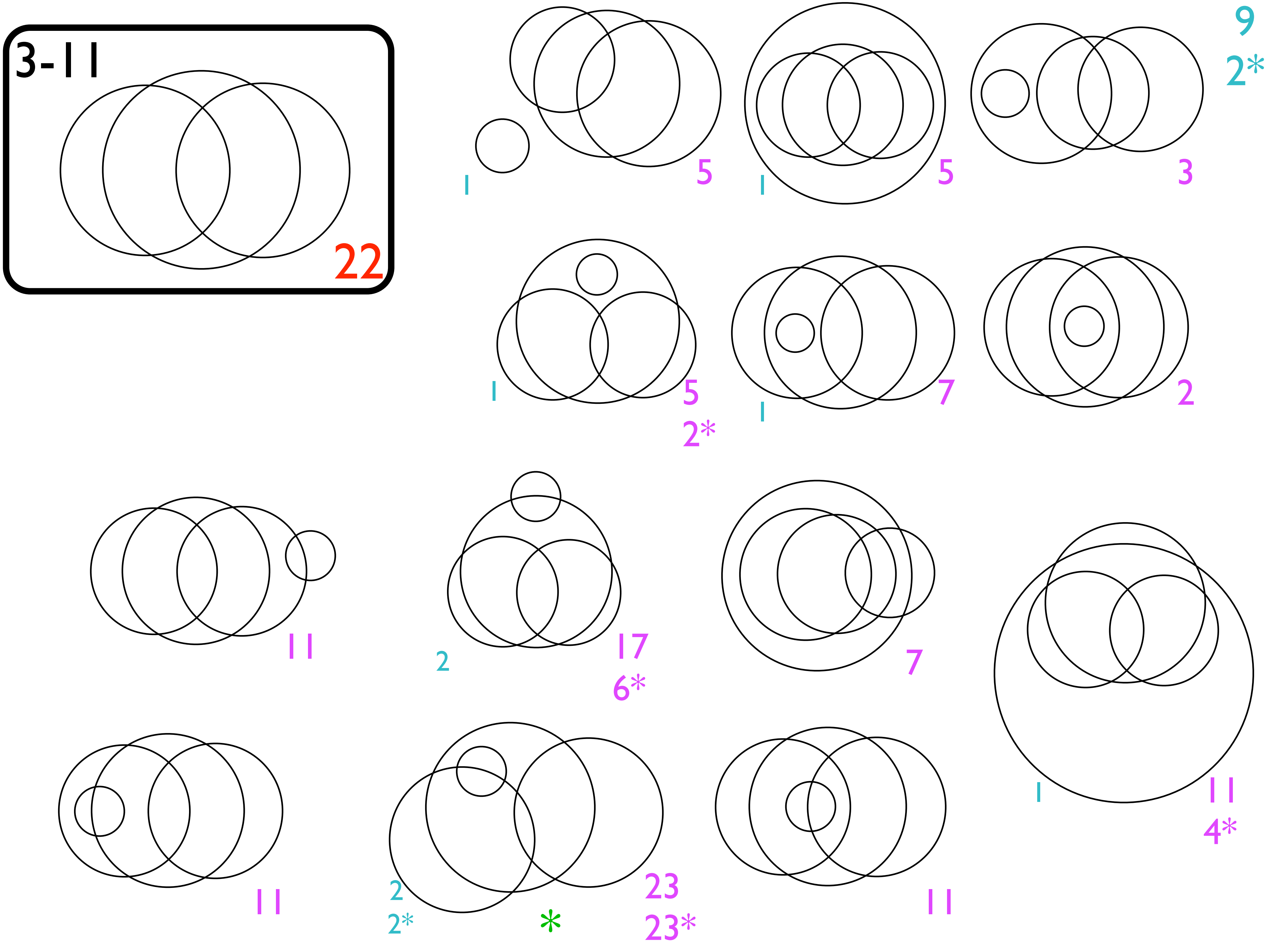
1 9,9,10,10 9



4,10,10,10 5



10,10,10,10 5


3-11
22
9
2*
1
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1
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3
1
5
2*
1
7
2
11
2
17
6*
7
1
11
4*
11
2
2*
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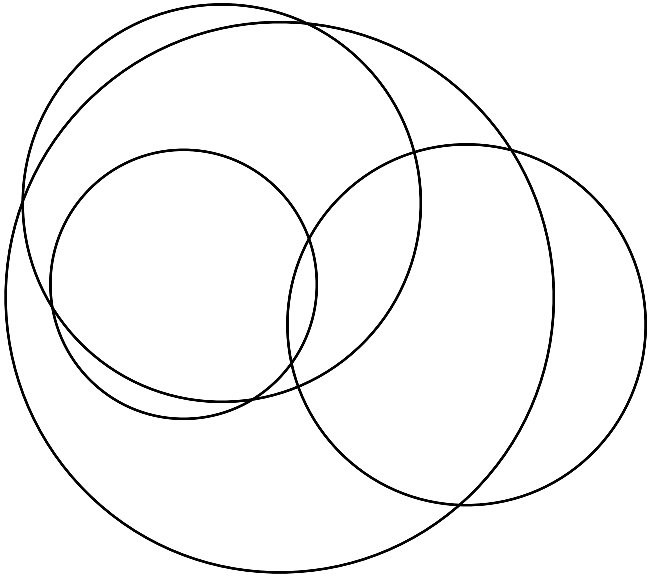
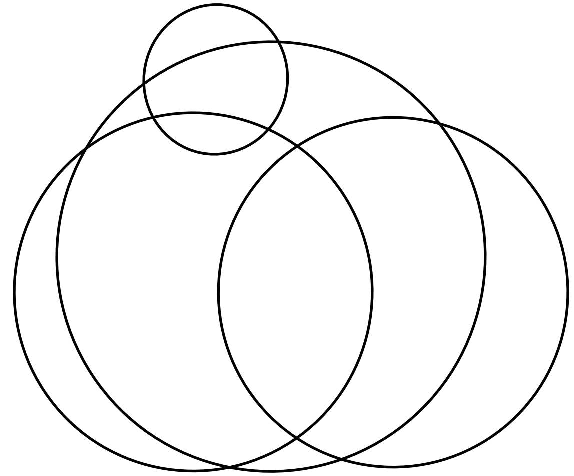
23
23*
11

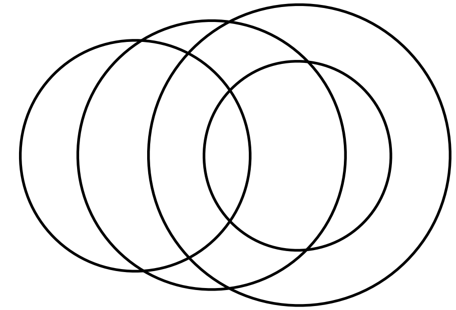
2
2*



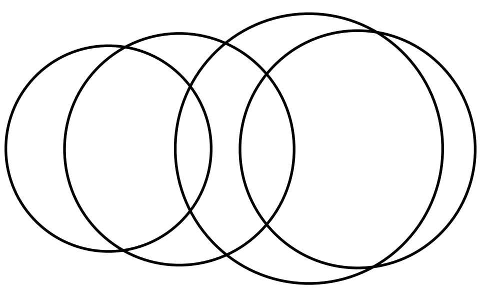
10,10,11,11* 15
15*



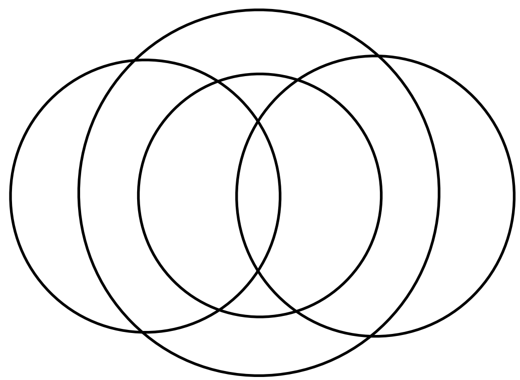
1 9,9,11,11* 31
1* 31*



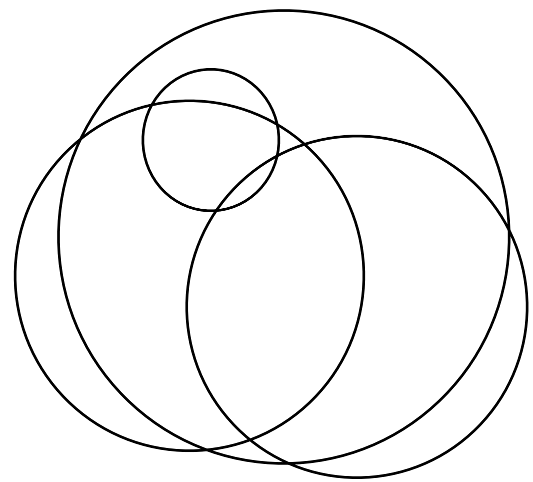
10,10,11,11 7



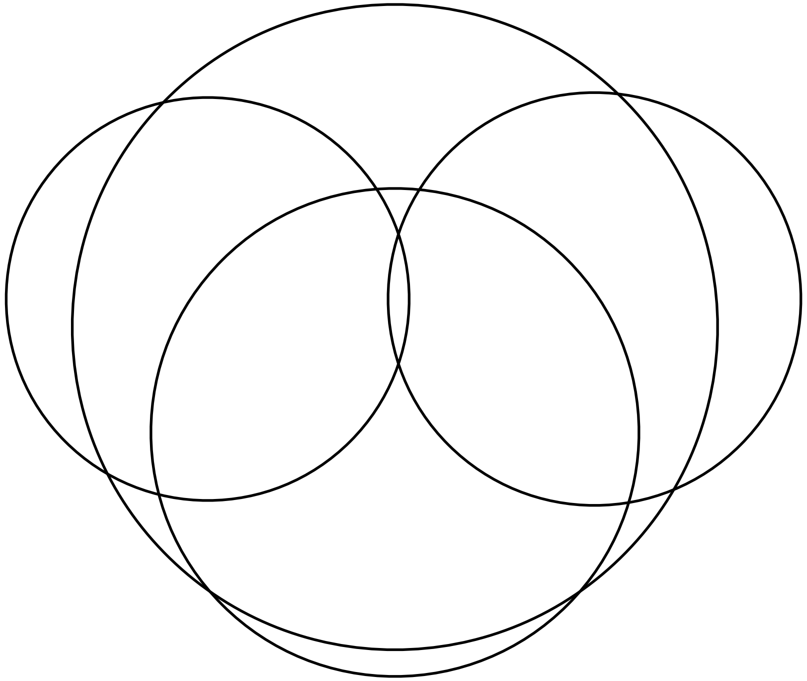
9,9,11,11 15



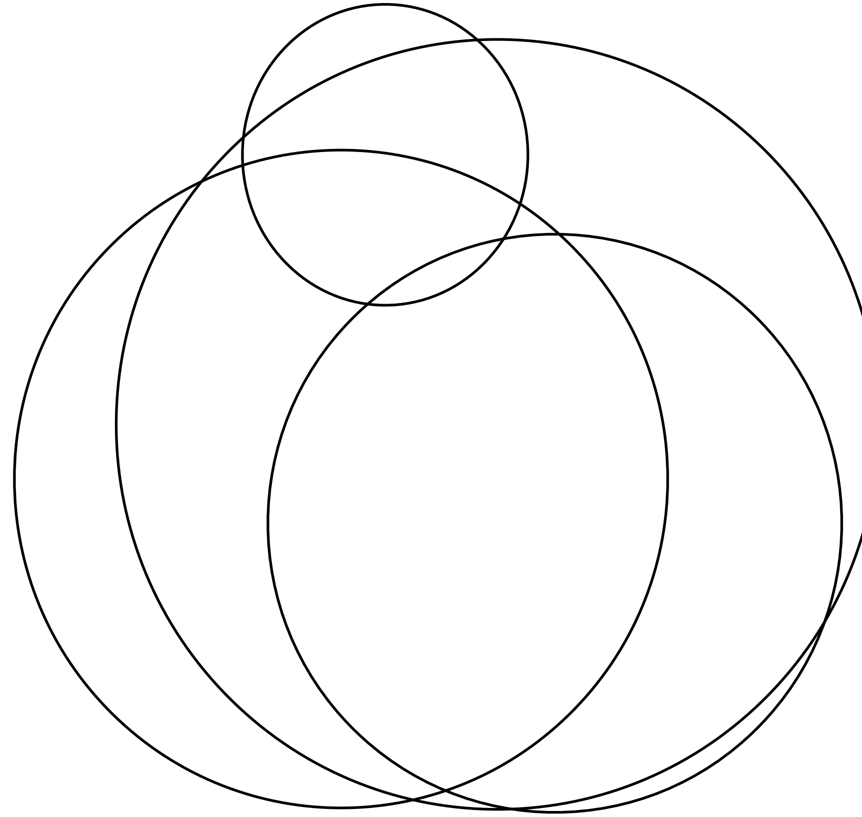
10,10,11,11 7



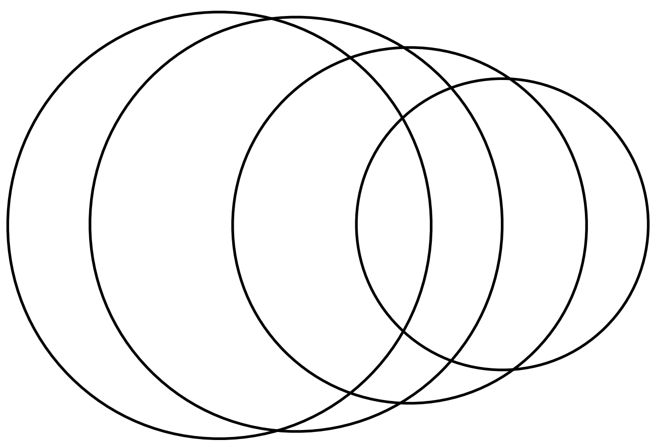
1 10,10,11,11* 15
1* 15*



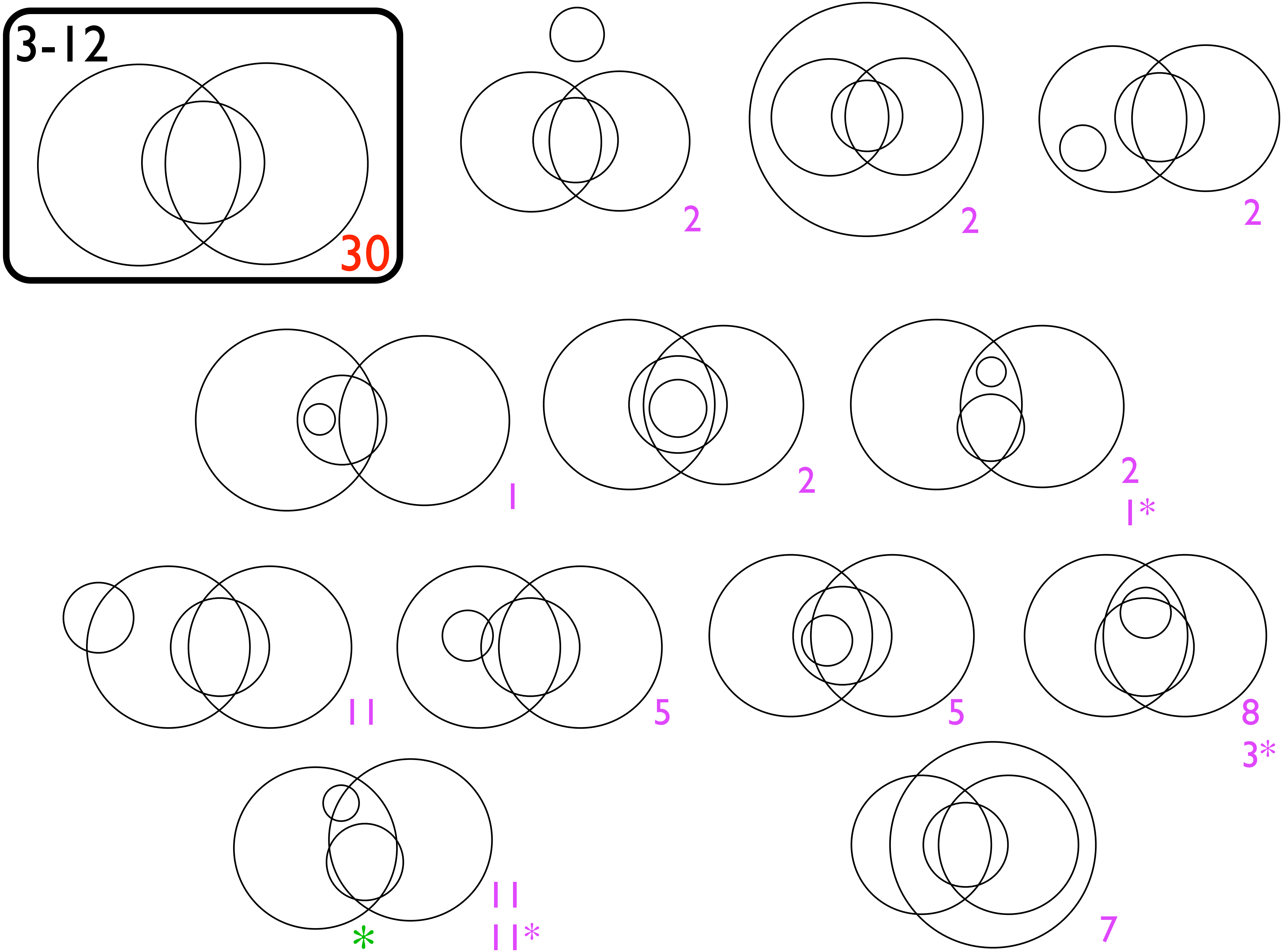
11,11,11,11 11
4*



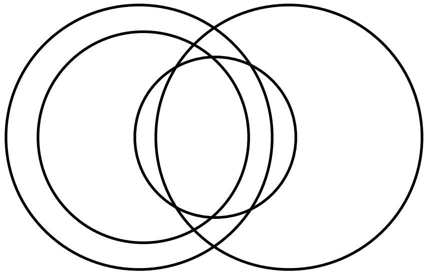
11,11,11,11* 15
15*



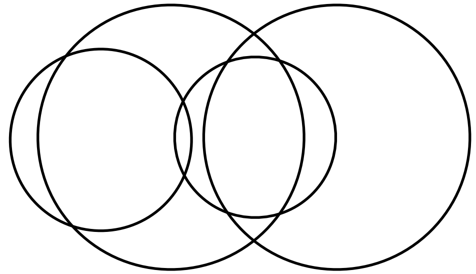
11,11,11,11 7


3-12
30
2
2
2
1
2
2
1*
11
5
5
8
3*
*
11
11*
7

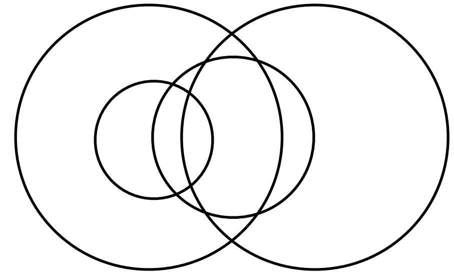
Ones where the new circle cuts two of the circles, including the centre one:



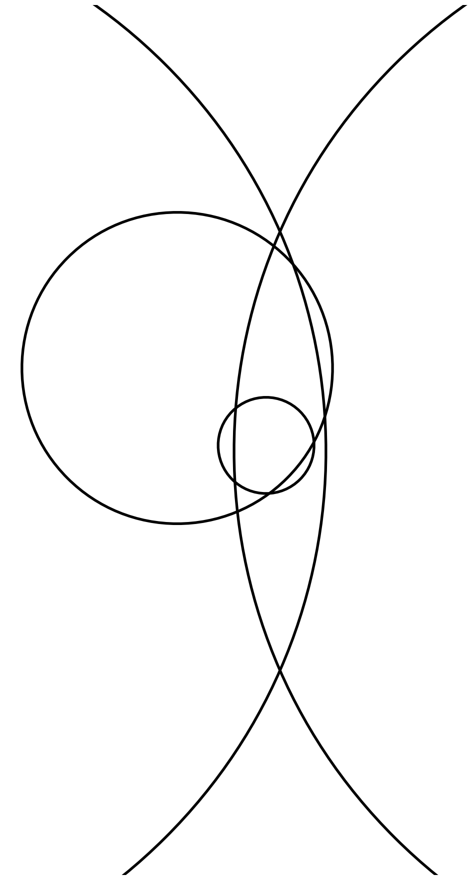
10,10,12,12 3

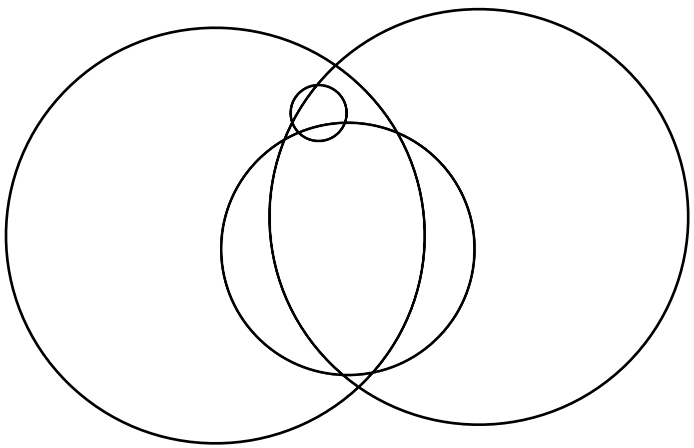


9,9,11,12 7



10,10,11,12 7

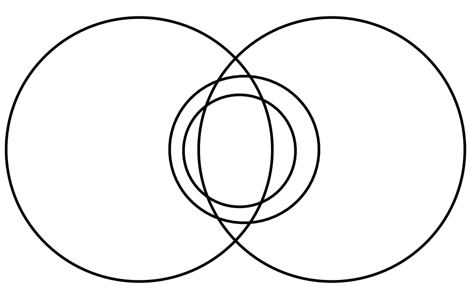




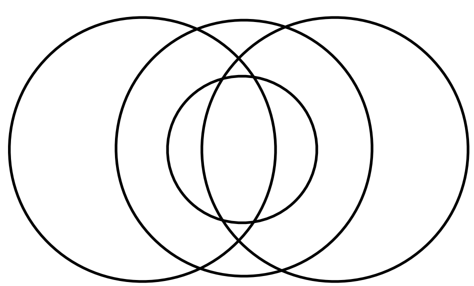
10,10,11,12 * 15
15*

10,10,12,12 * 7
7*

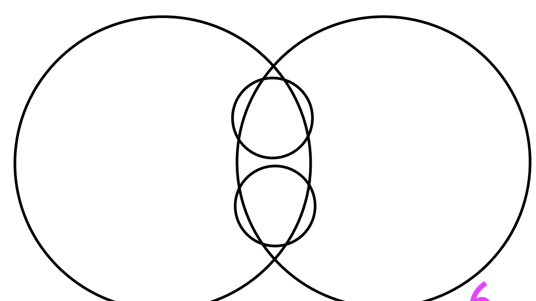
Ones where the new circle cuts two of the circles, not including the centre one:



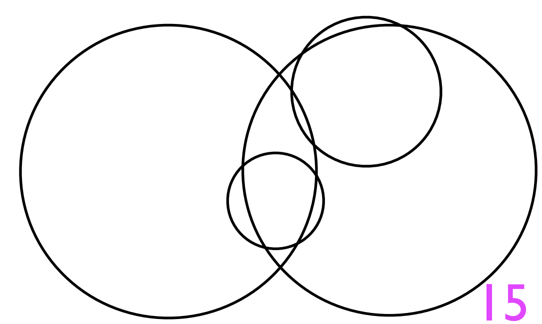
10,10,12,12 2



10,10,11,12 2

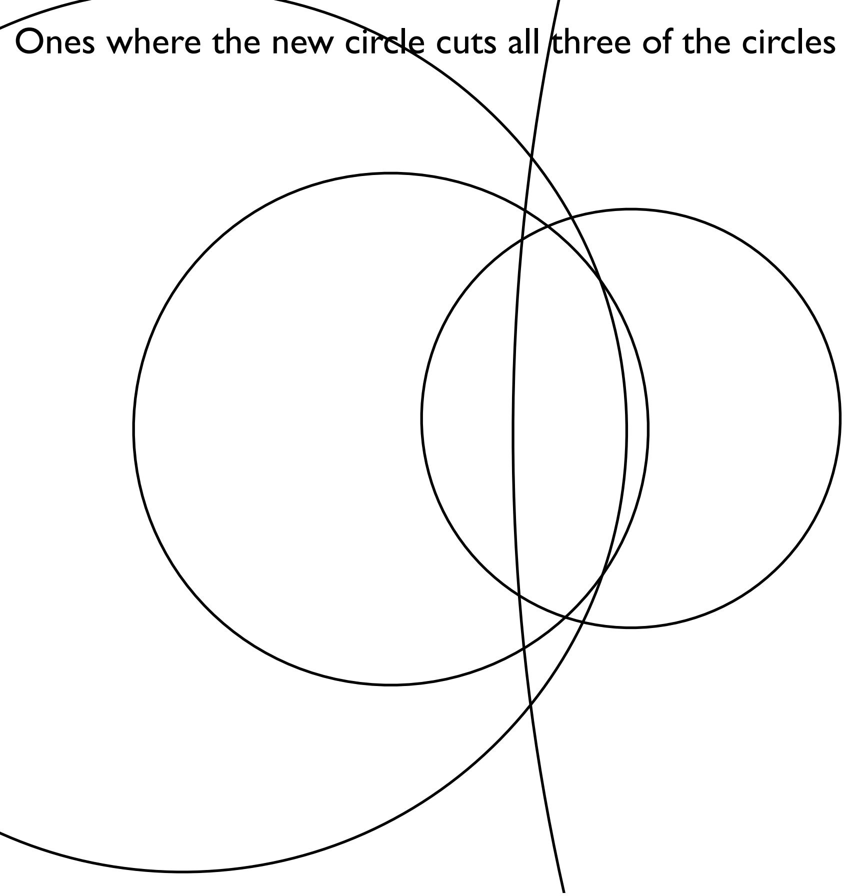


9,9,12,12 6
3*

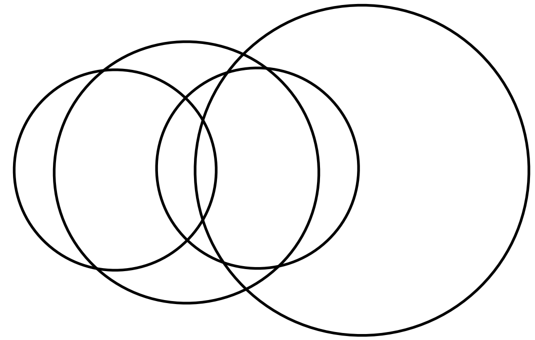


9,9,11,12 * 15
15*

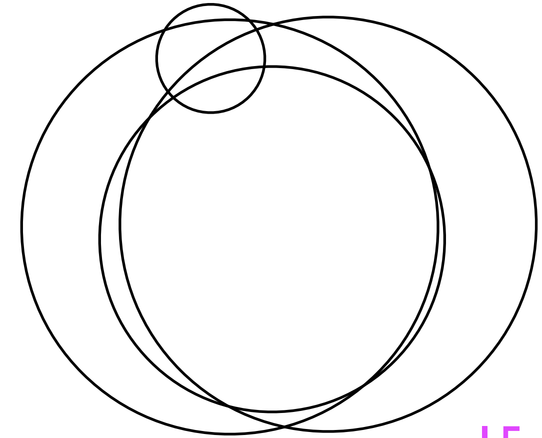
Ones where the new circle cuts all three of the circles



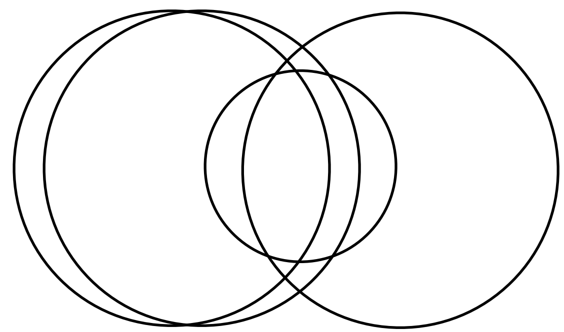
12,12,12,12 3



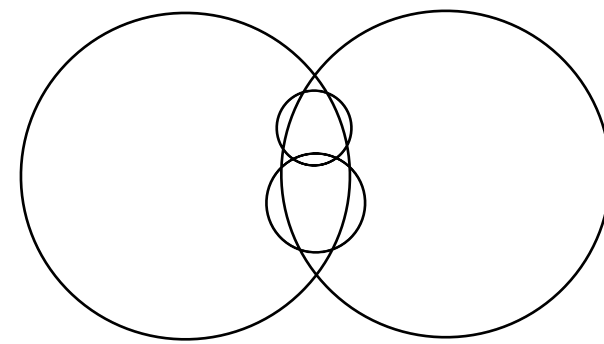
11,11,11,12 7



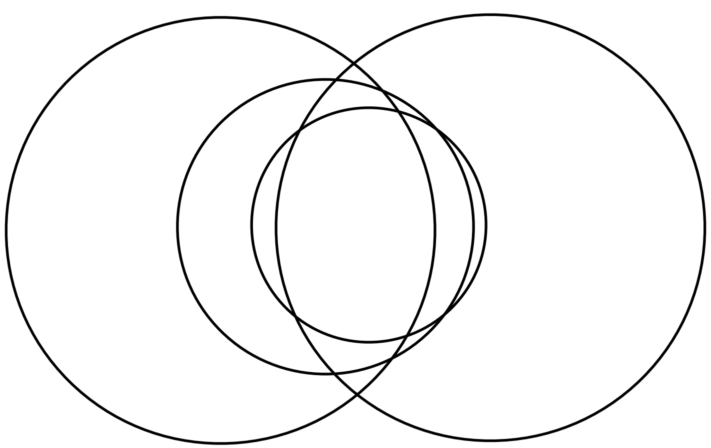
11,11,11,12 * 15 15*



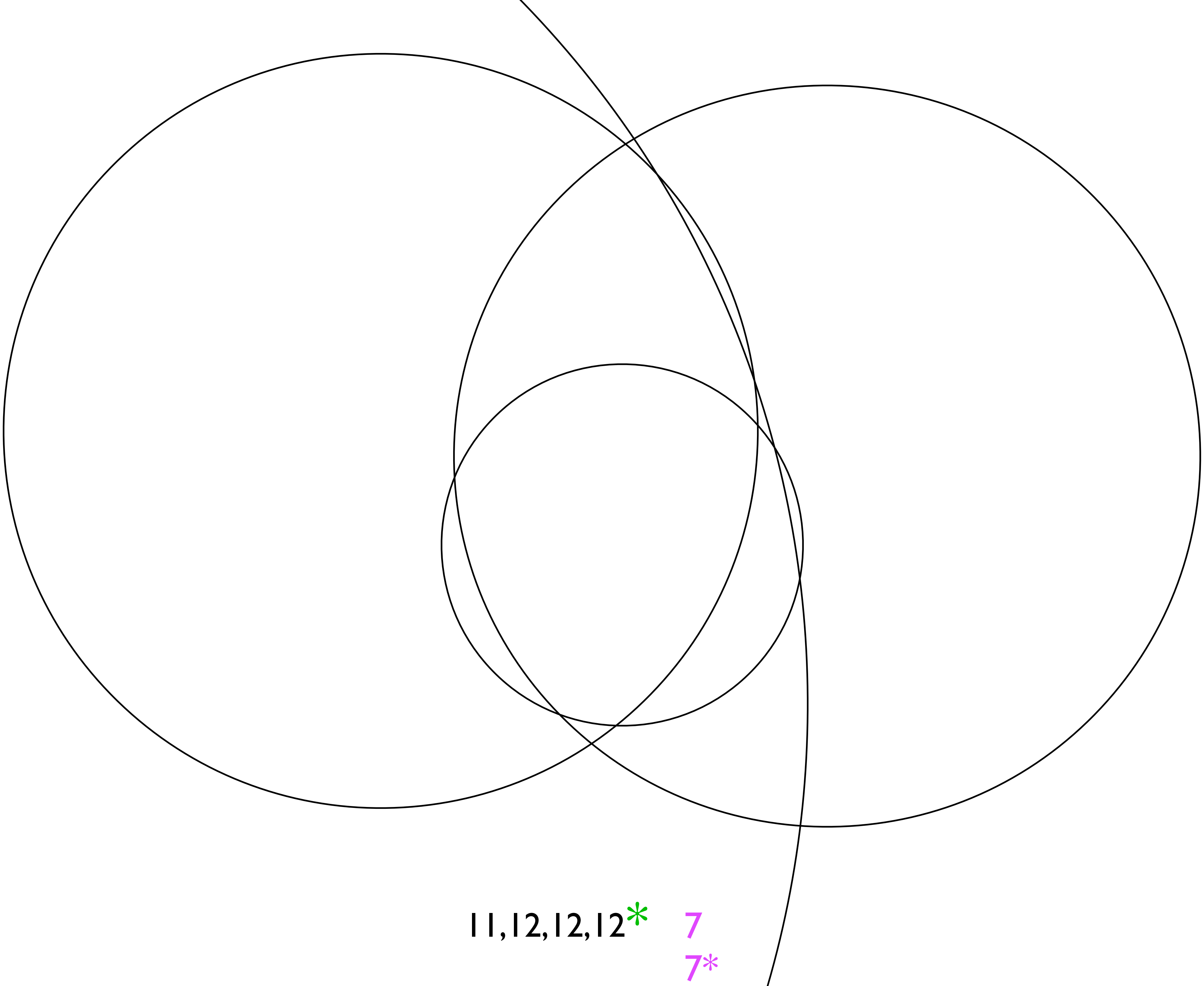
11,11,12,12 7

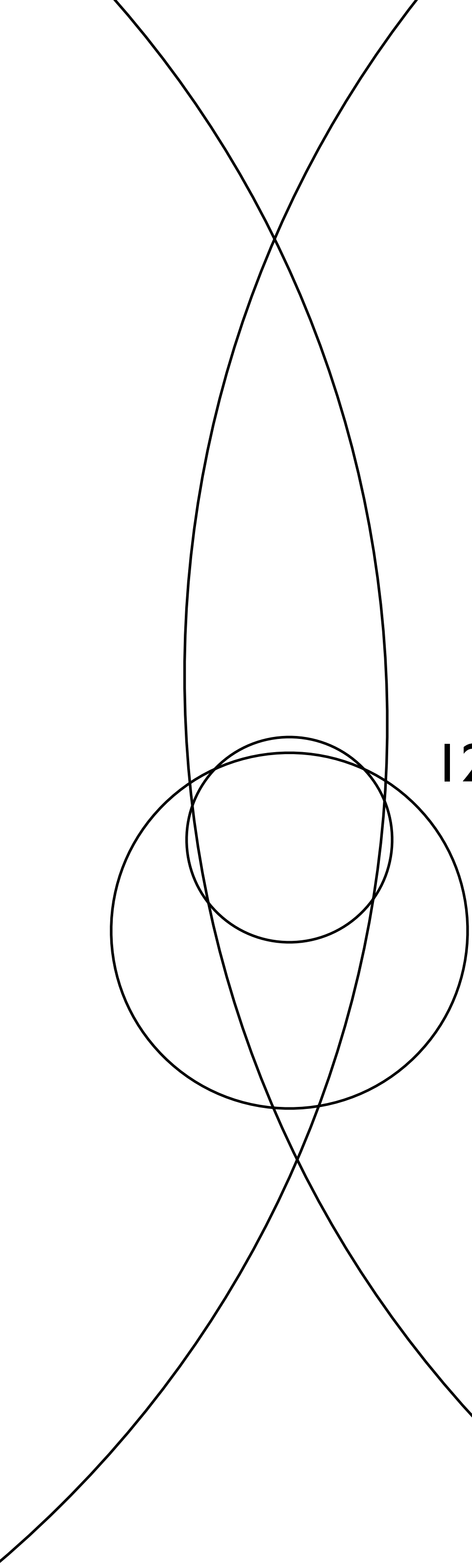


11,11,12,12 13 6*



11,12,12,12 3

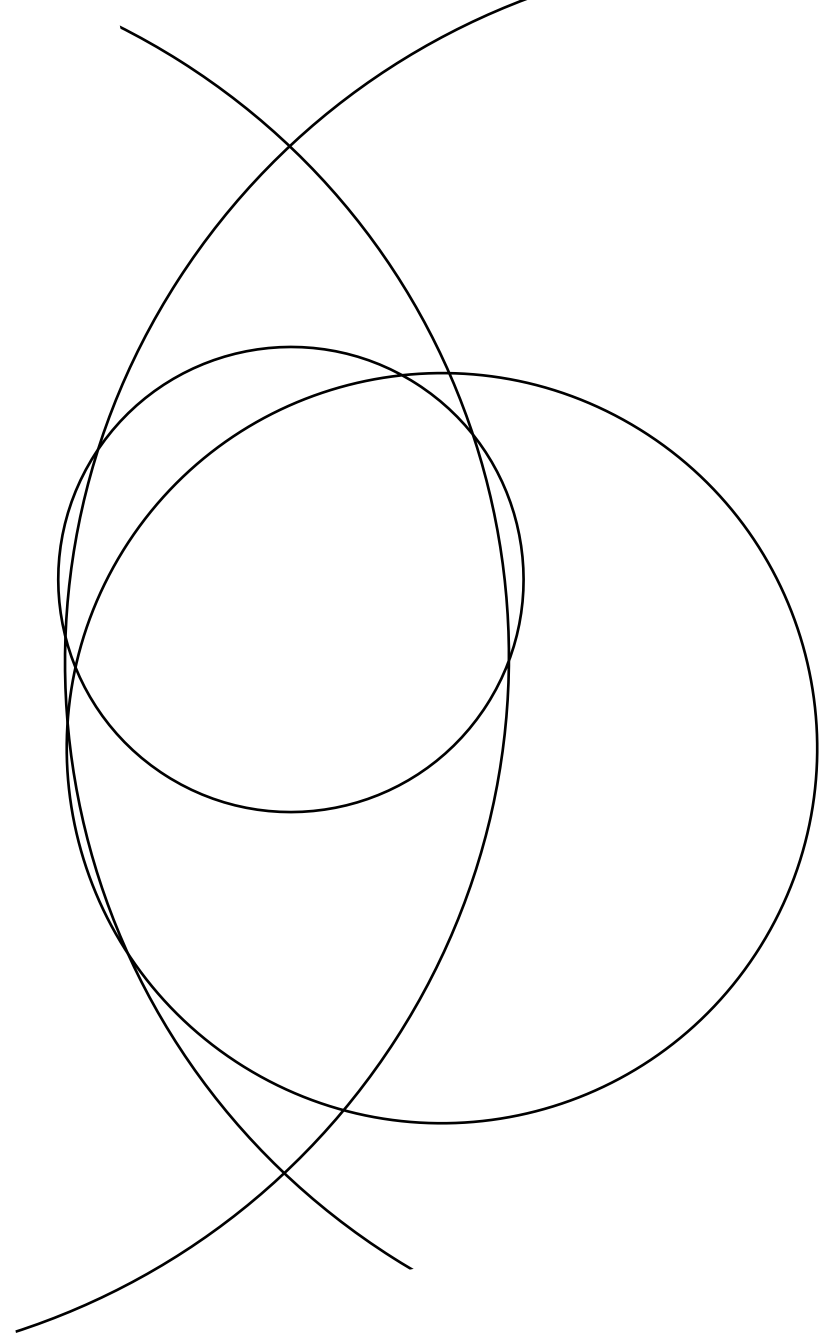

11,12,12,12*
7
7*



12,12,12,12

5

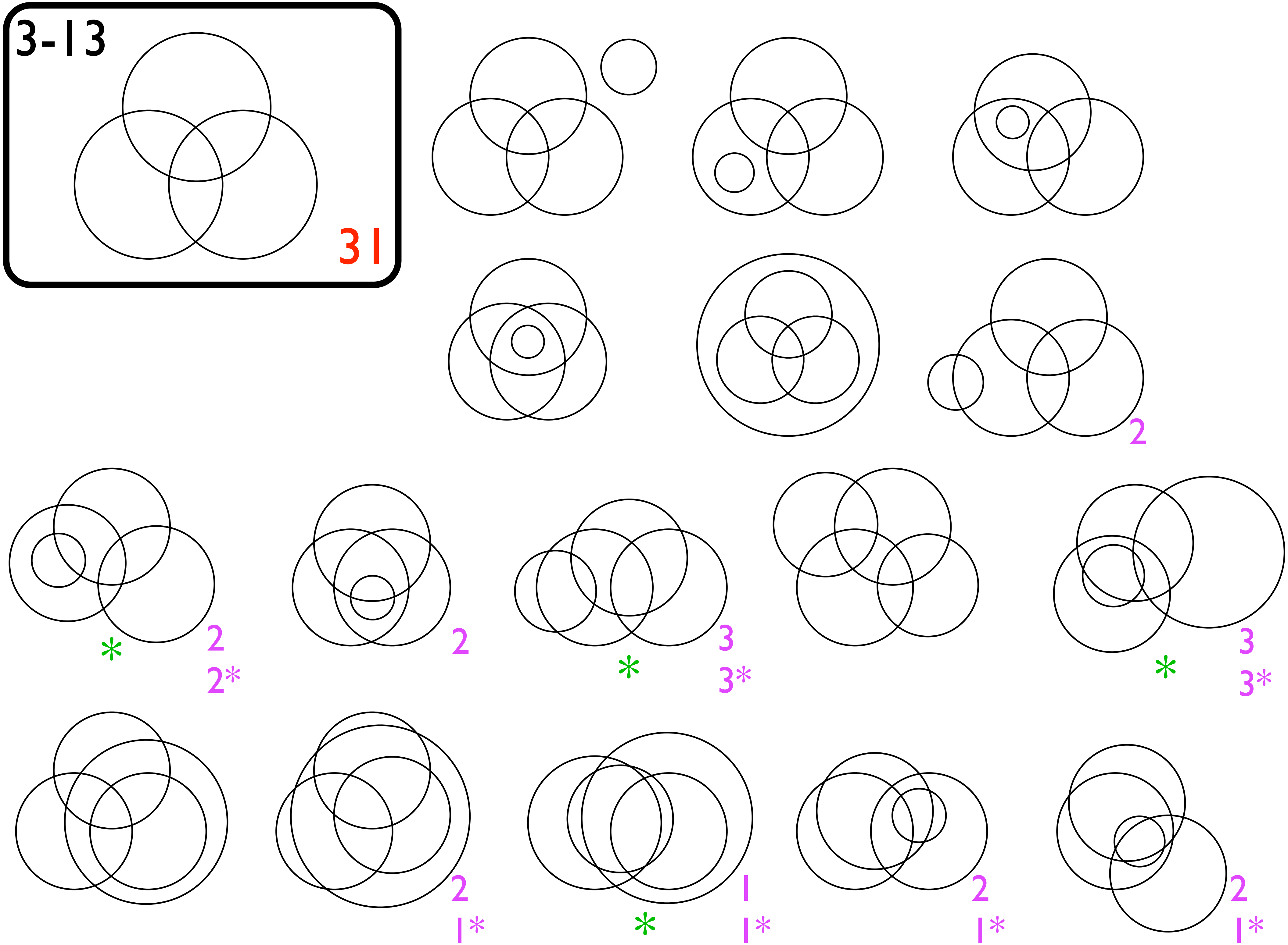
2*



12,12,12,12*

7

7*


3-13
31
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3
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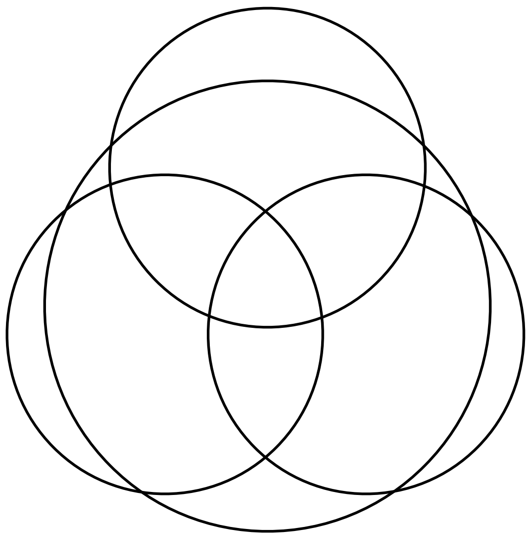
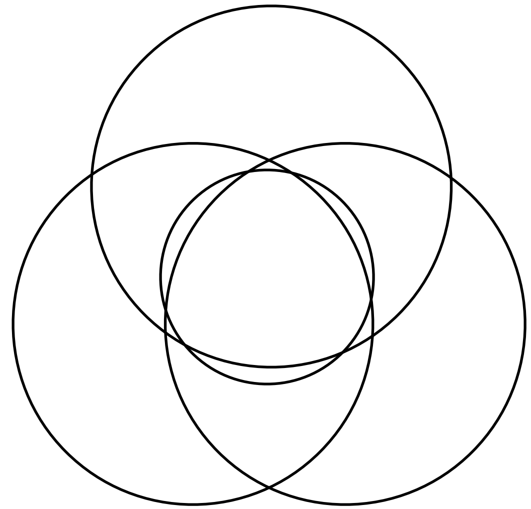
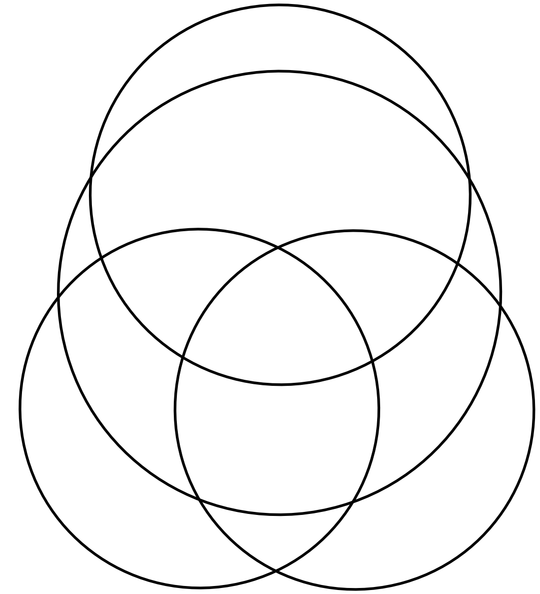
3*
2
1*
*
1
1*
2
1*
2
1*



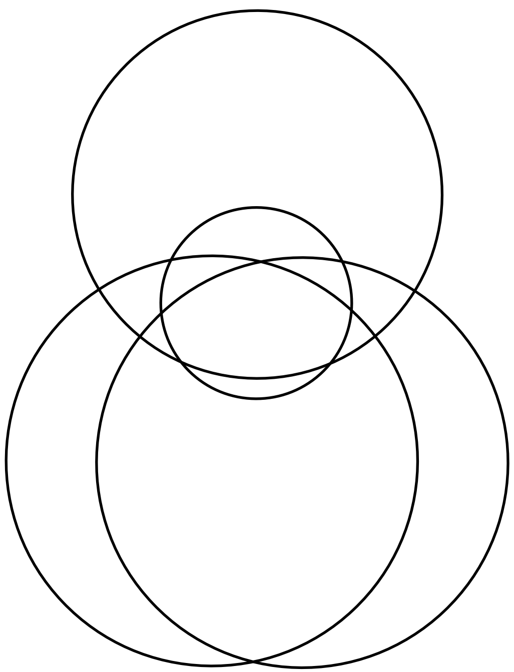
11,11,11,13 3



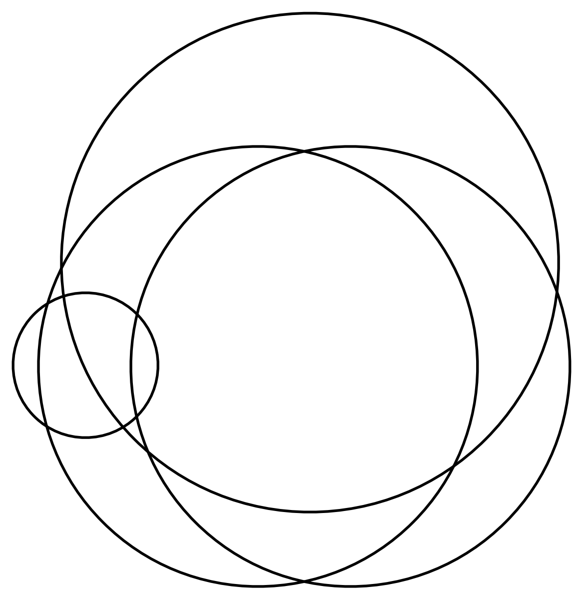
12,12,12,13 3



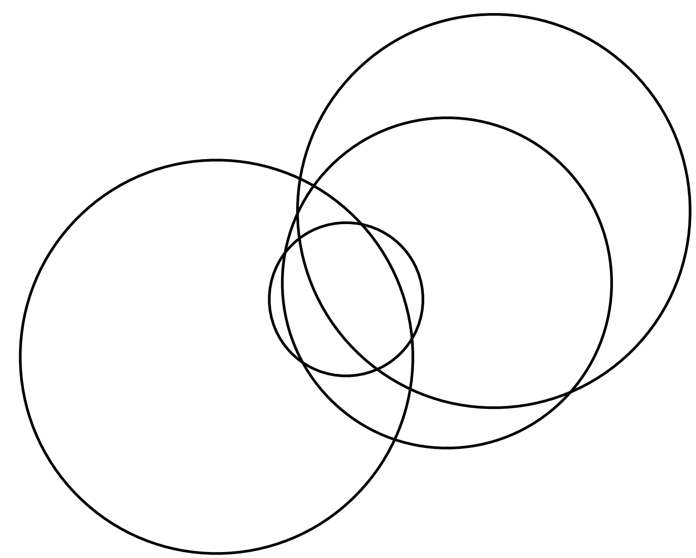
11,11,13,13 1



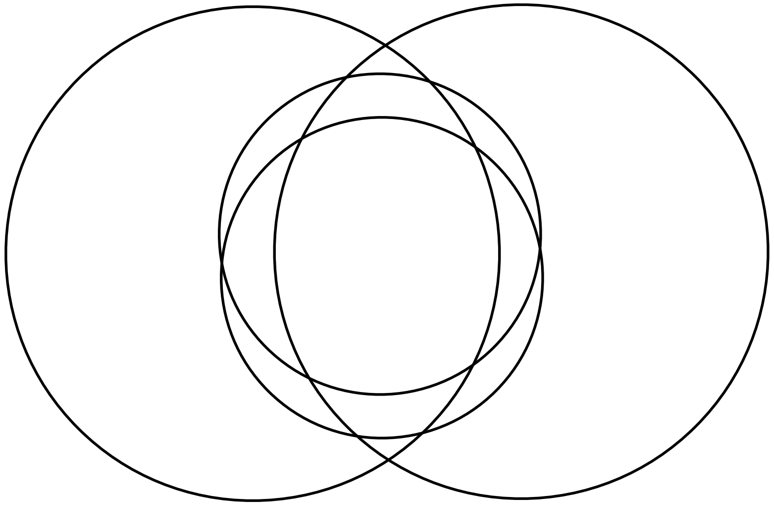
12,12,13,13 1



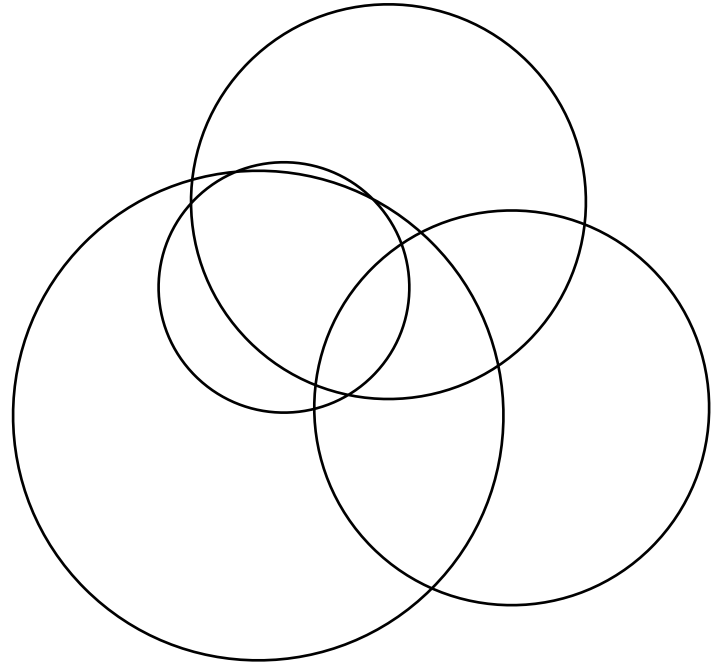
11,11,11,13 * 3 3*



11,12,12,13* 3 3*



12,12,13,13



11,12,13,13 * 1
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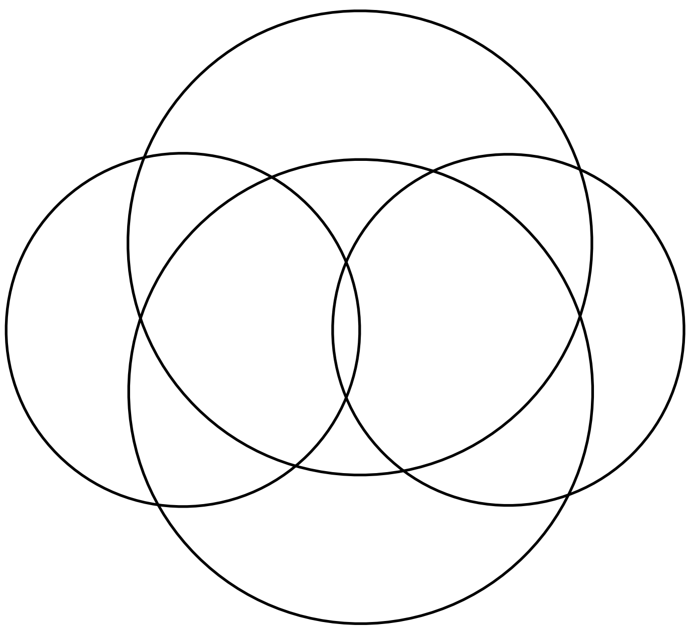
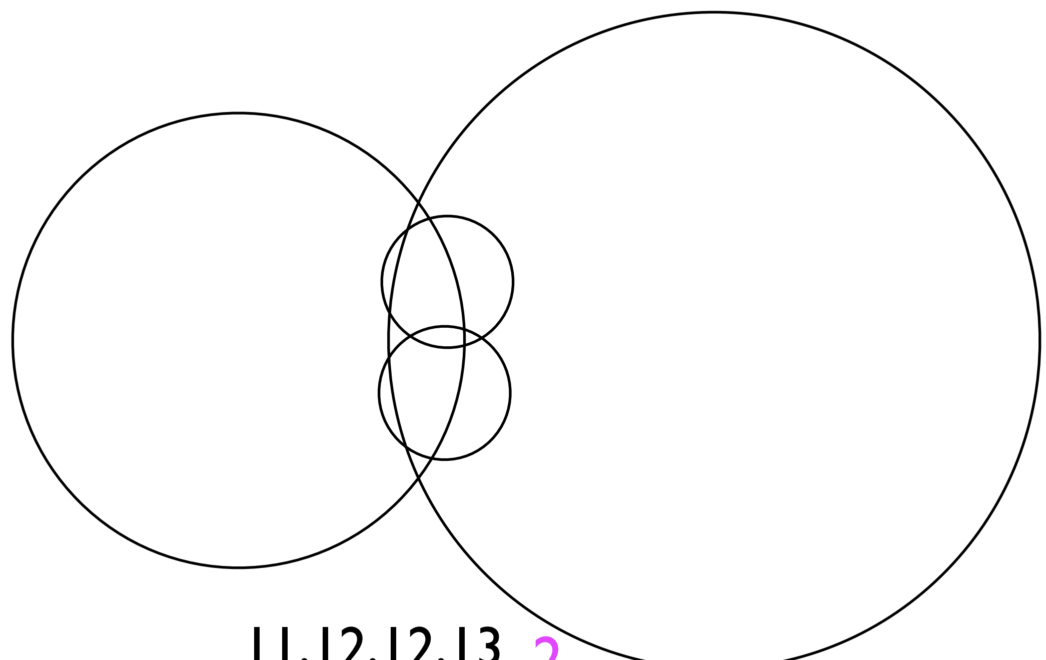


11,11,13,13 1



11,12,12,13 2
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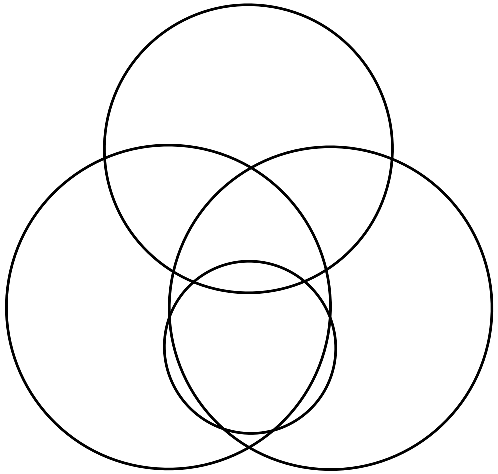

11,11,12,13 5
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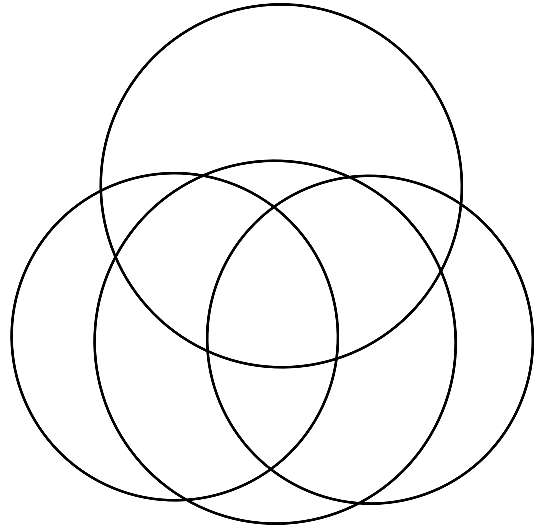
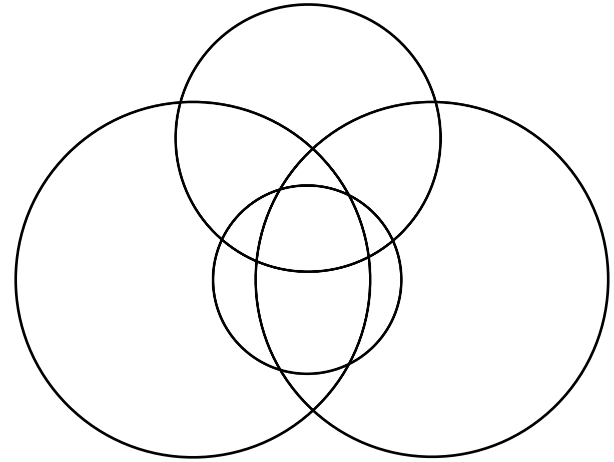
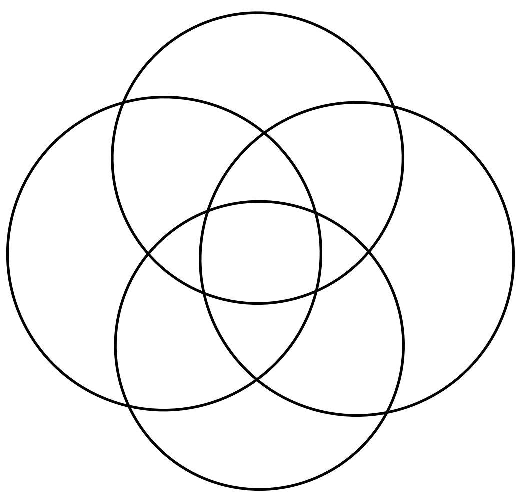


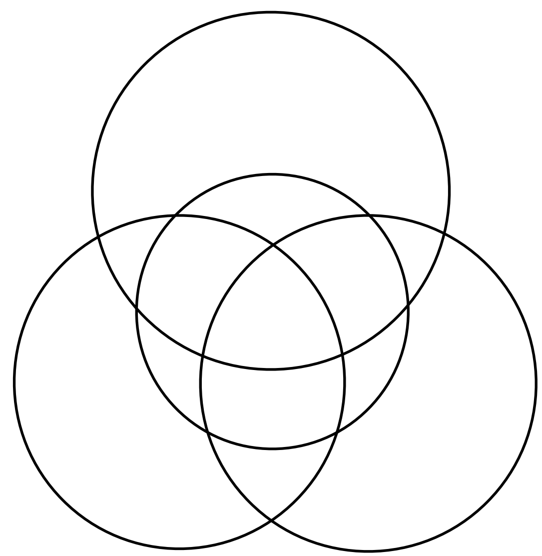
11,13,13,13



12,13,13,13



13,13,13,13



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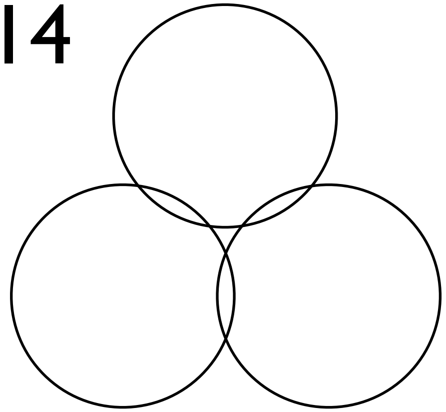

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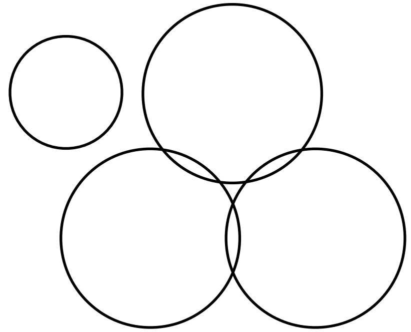

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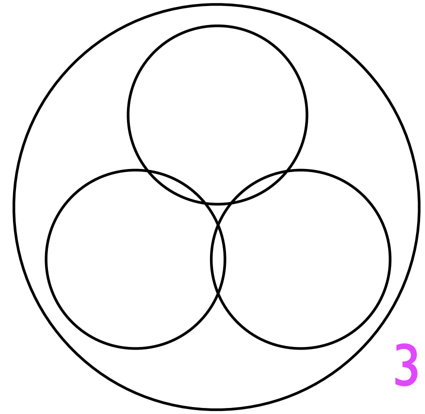

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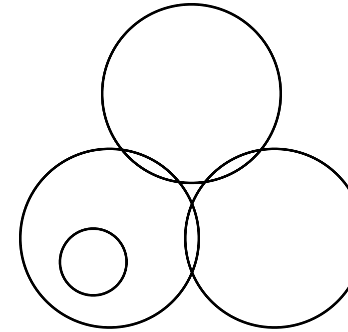

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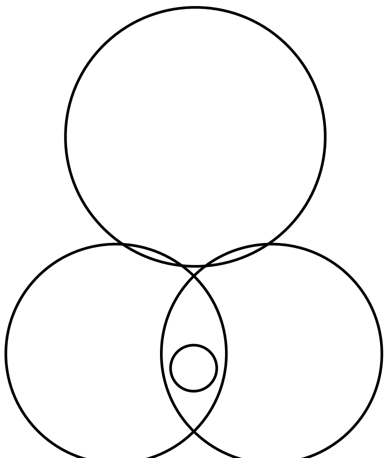
2*



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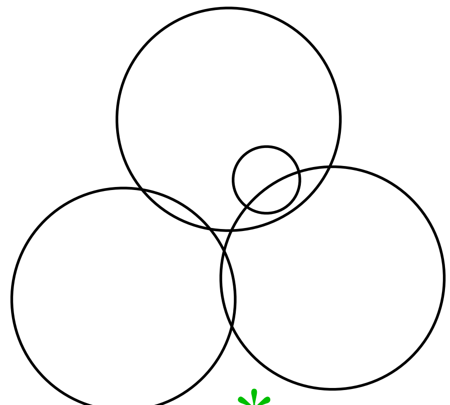

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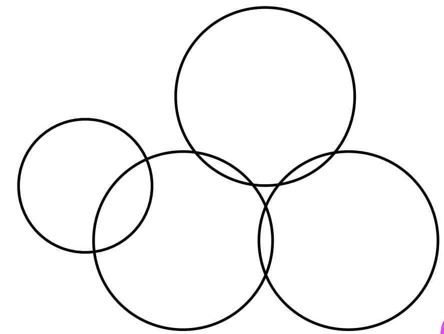
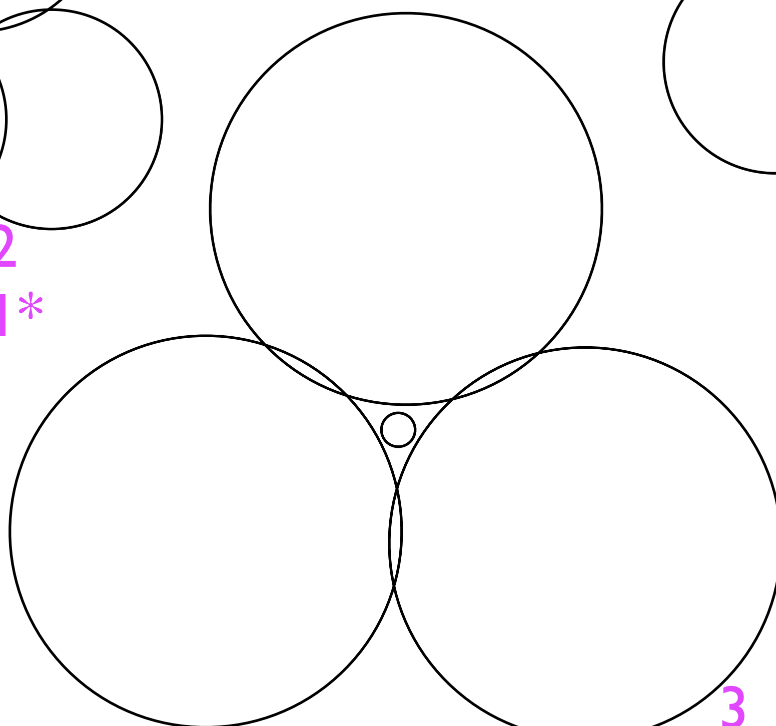


17
6*



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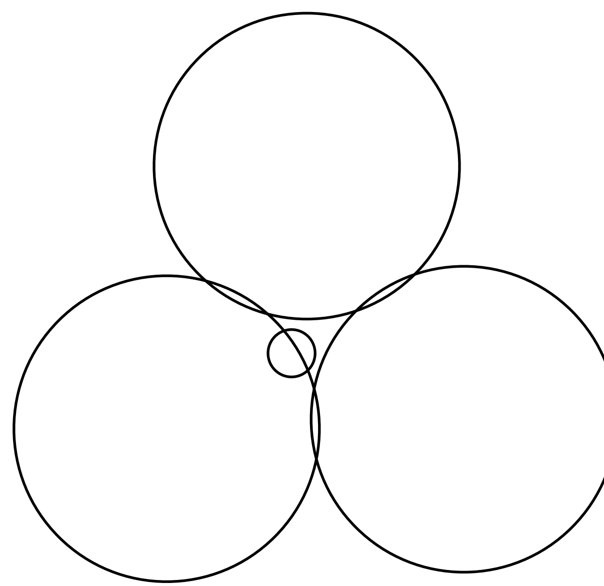

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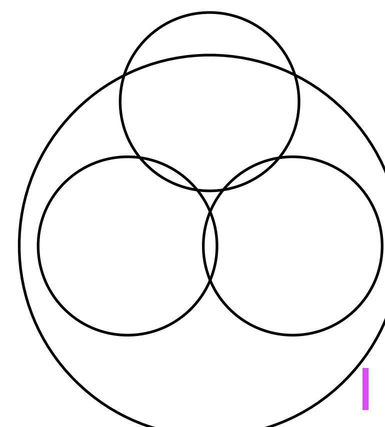

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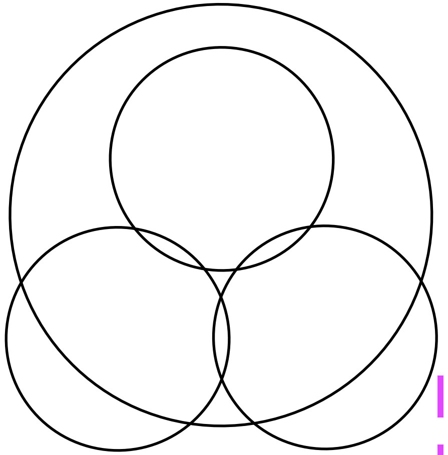
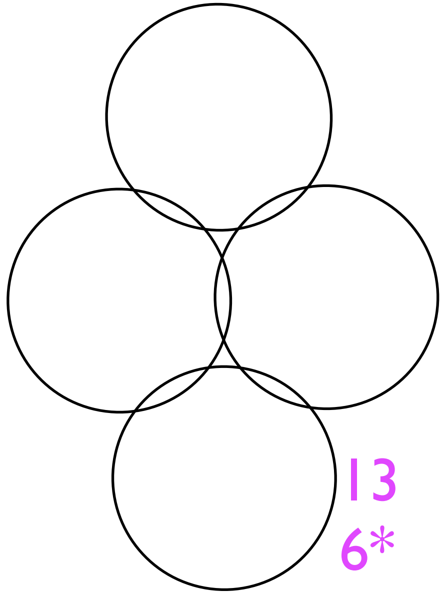


19
12*



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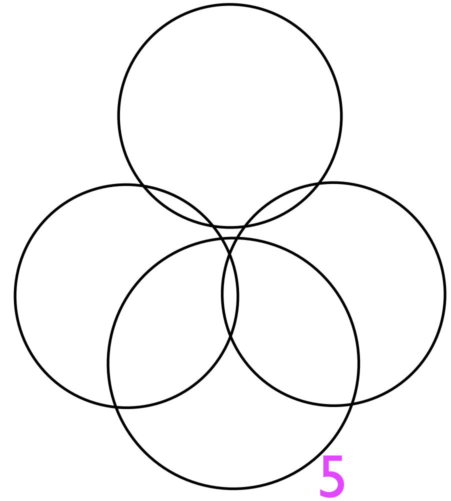
6*



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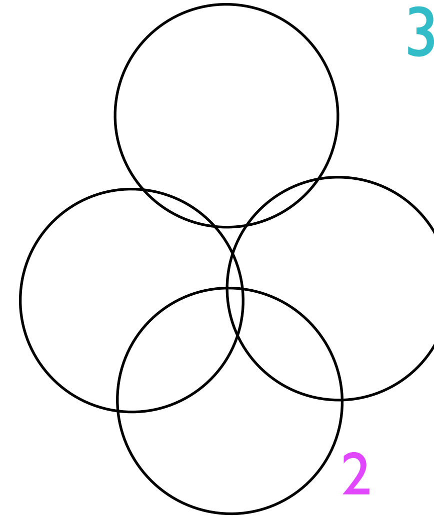
2*



2
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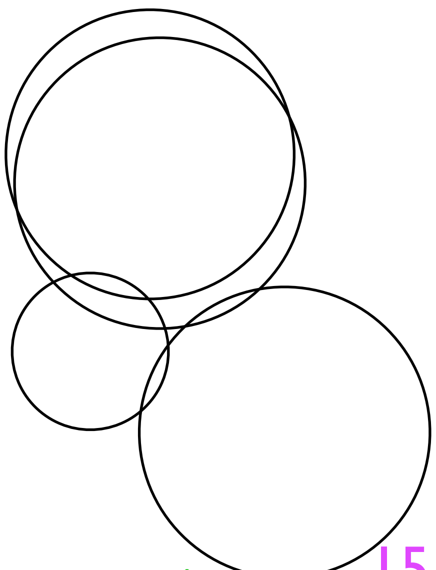

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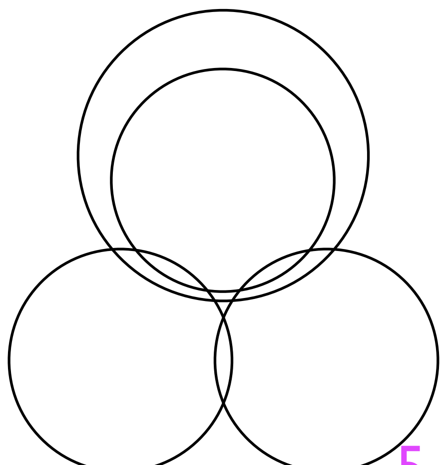
15
15*



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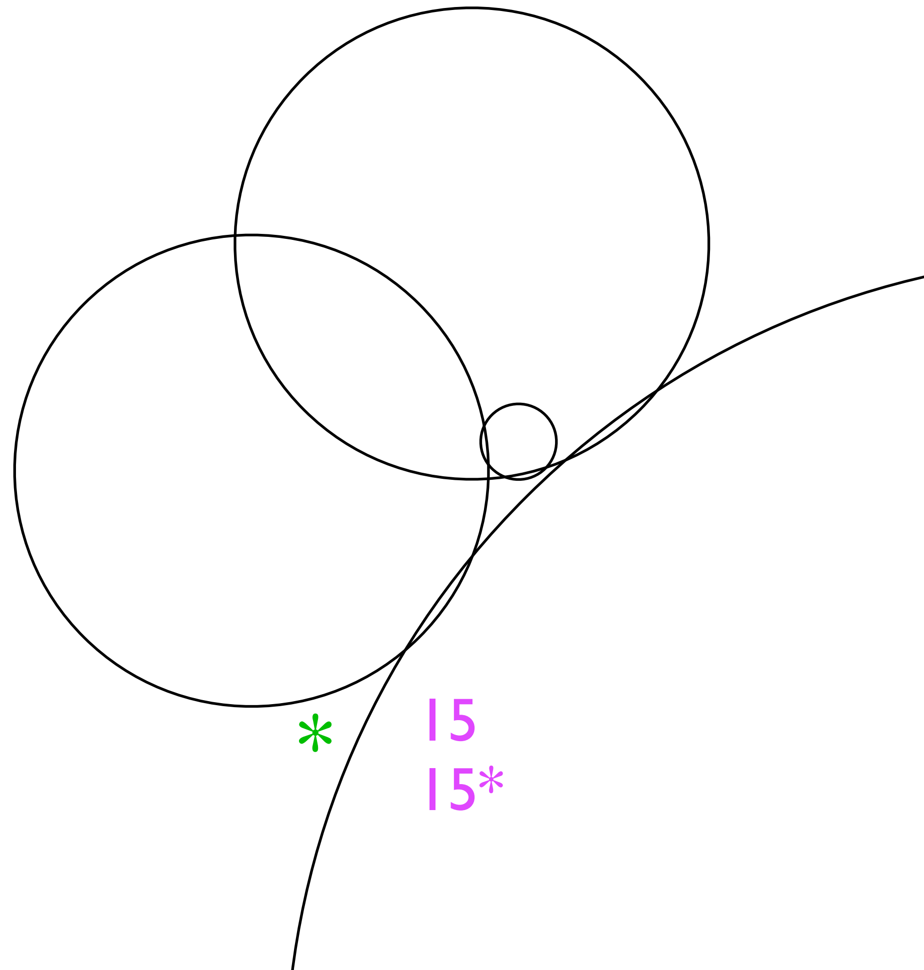
2*



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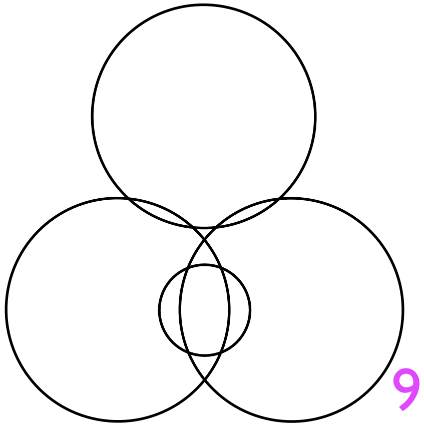
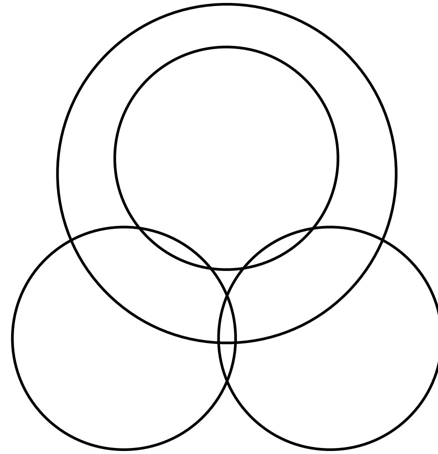
15
15*



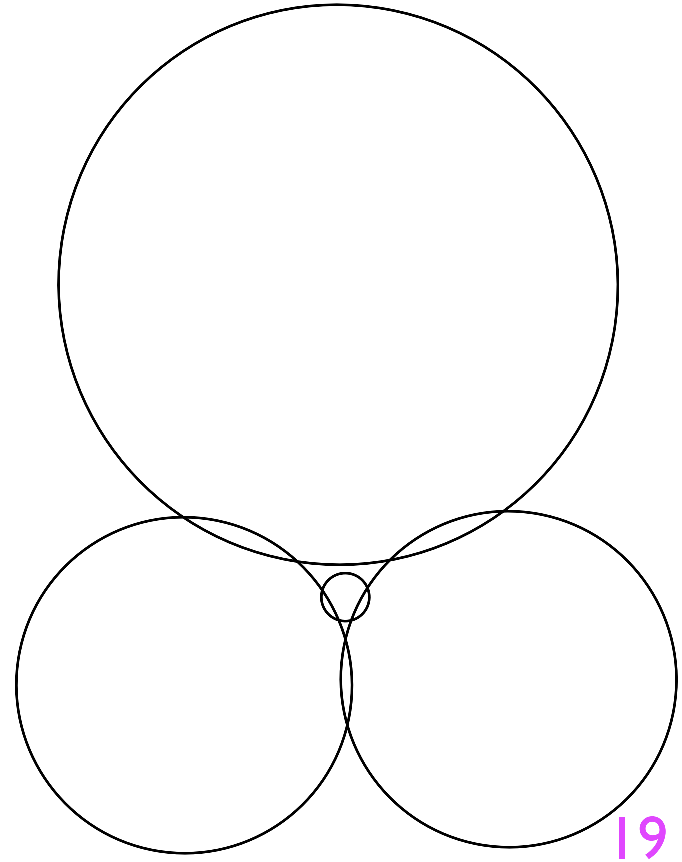
9

7*



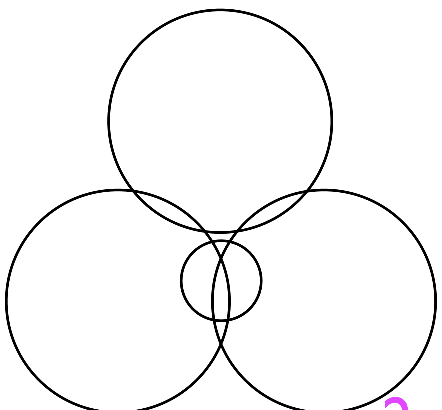
2

1*



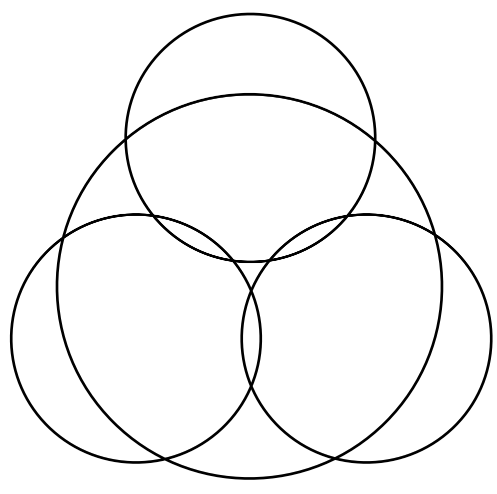
19

12*



2

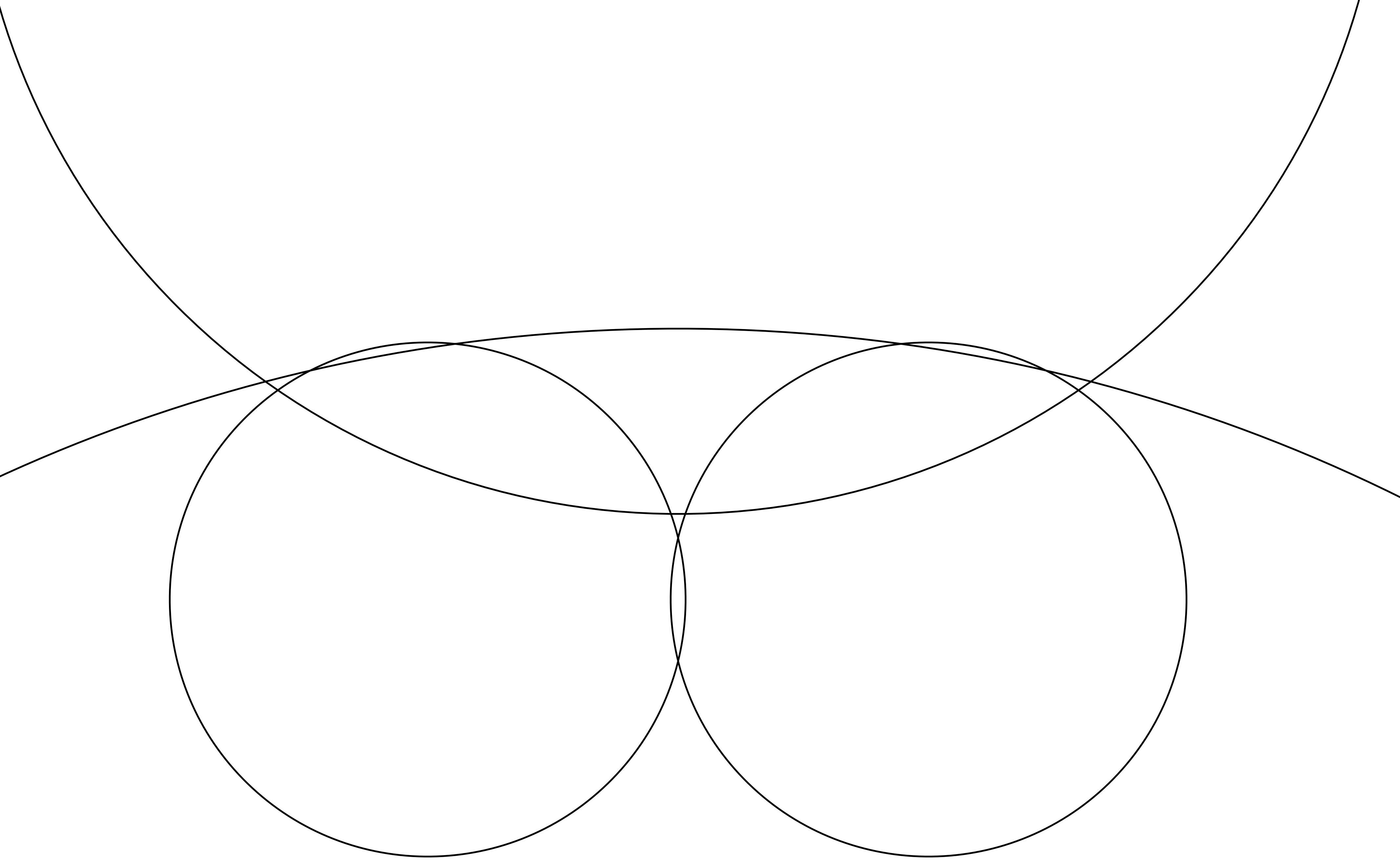
1*



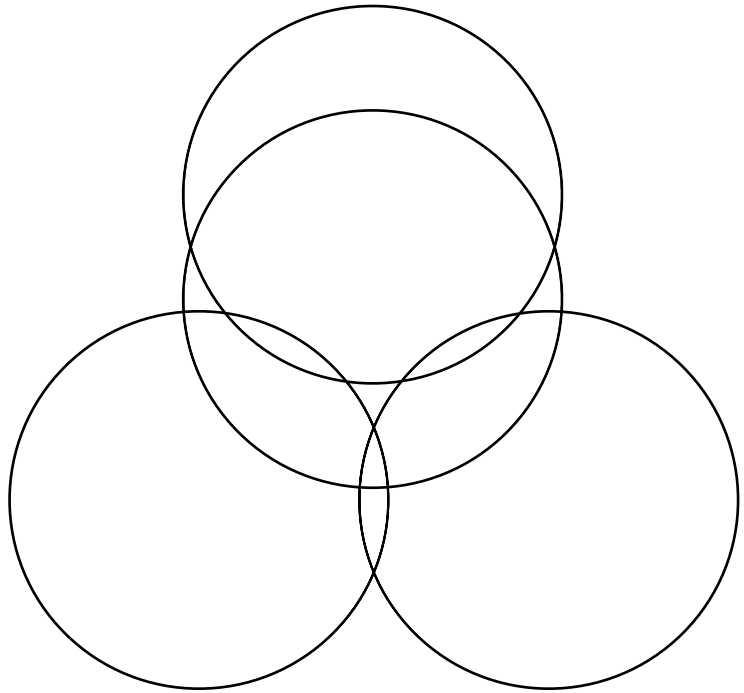
11,11,11,14

19

4*



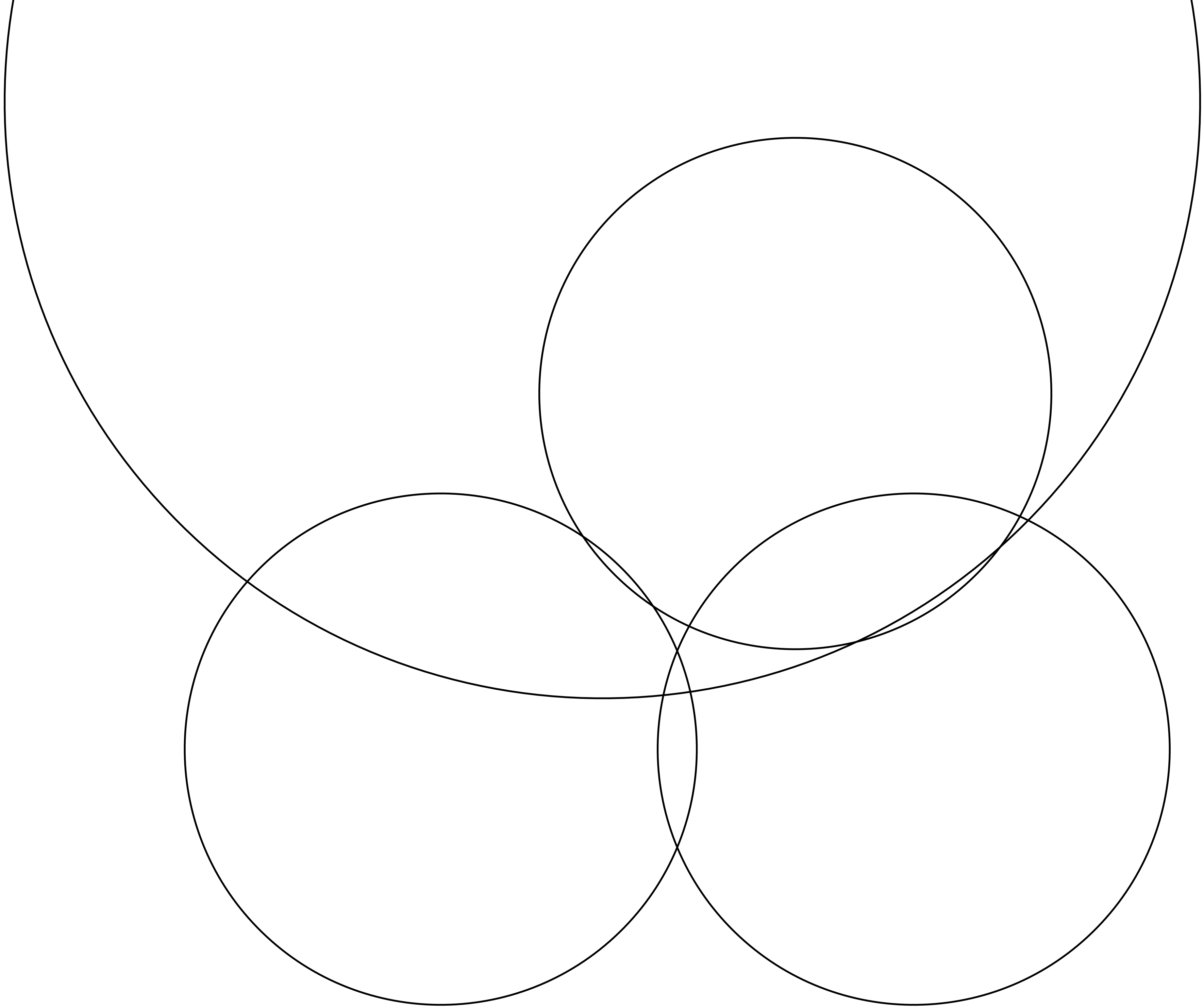
11,12,12,14 5
2*



11,11,13,14

5

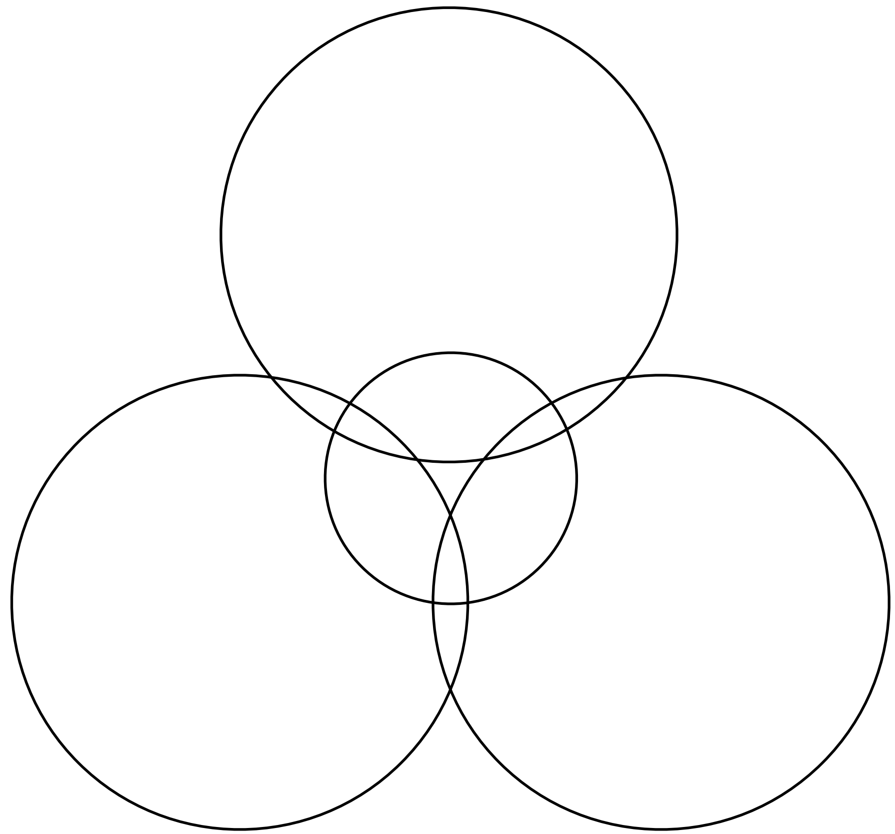
2*



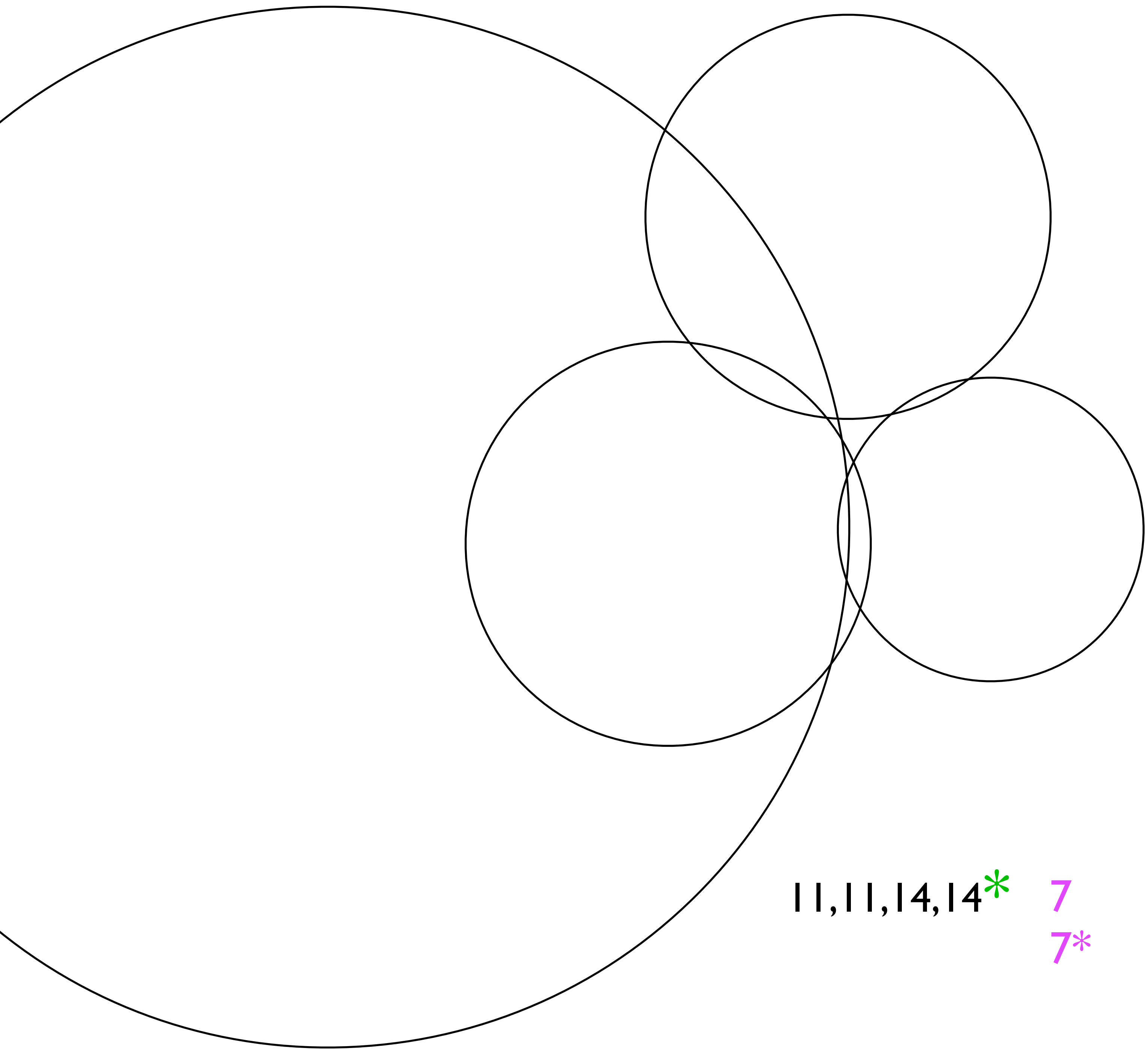
11,12,13,14*

3

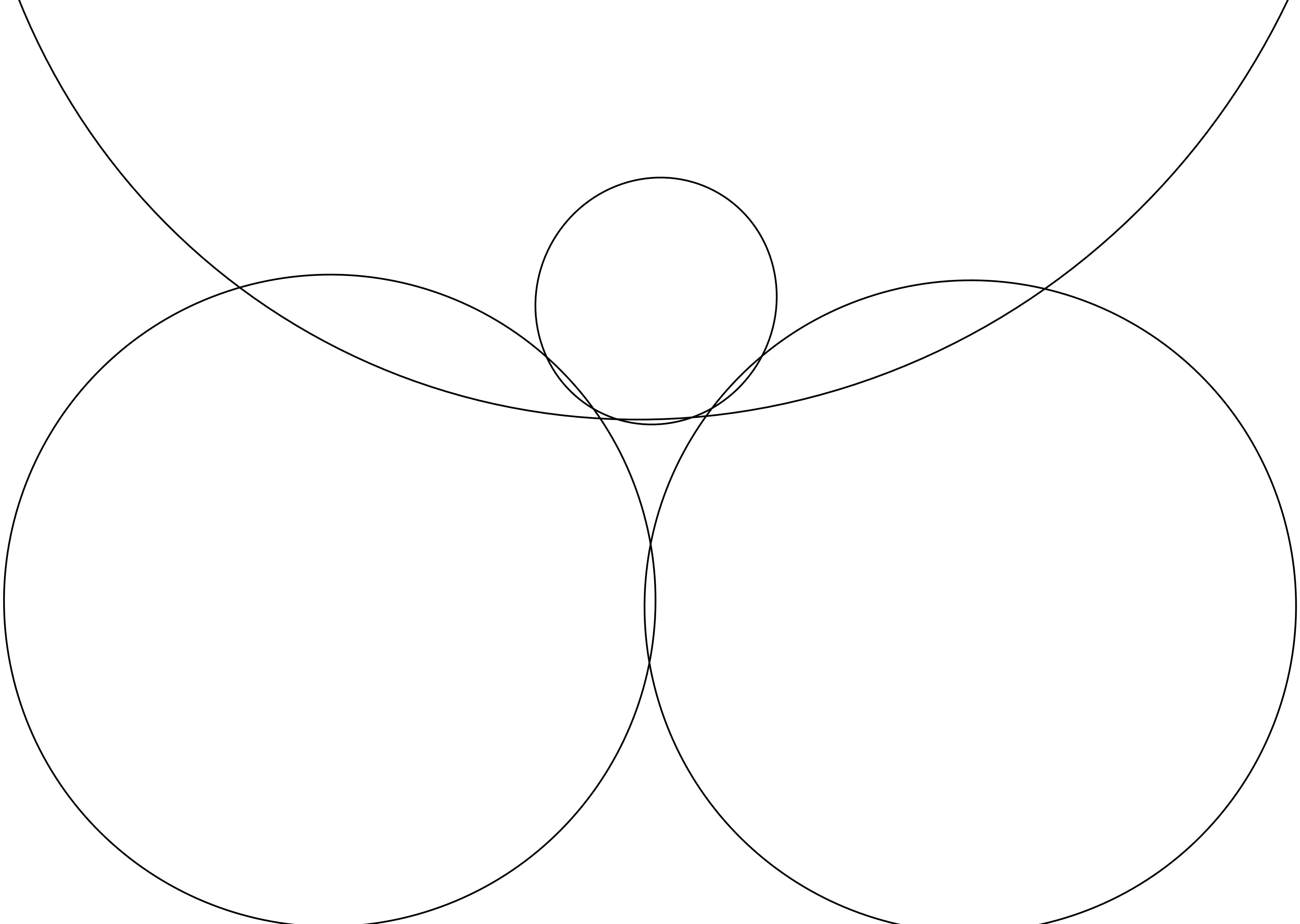
3*



13,13,13,14

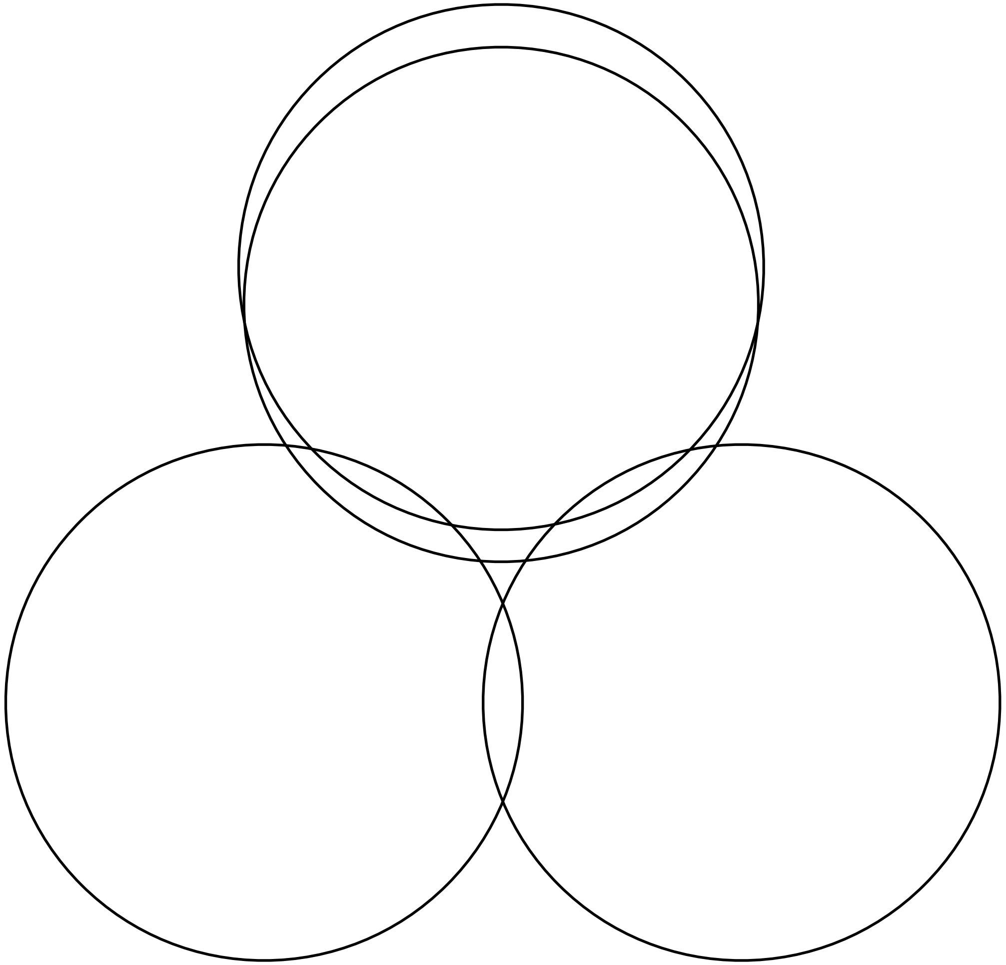


11,11,14,14*
7
7*



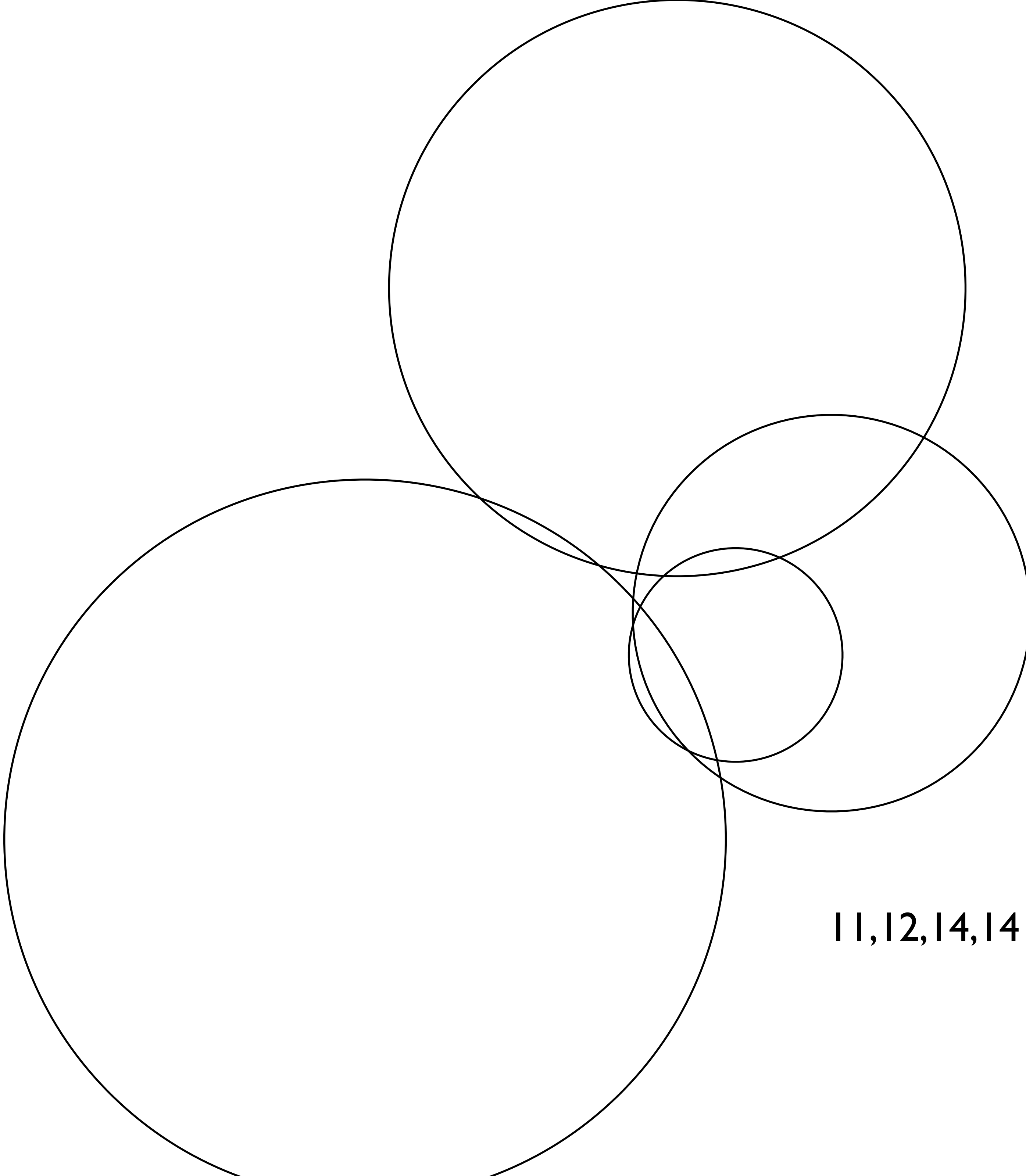
11,11,14,14 11

4*



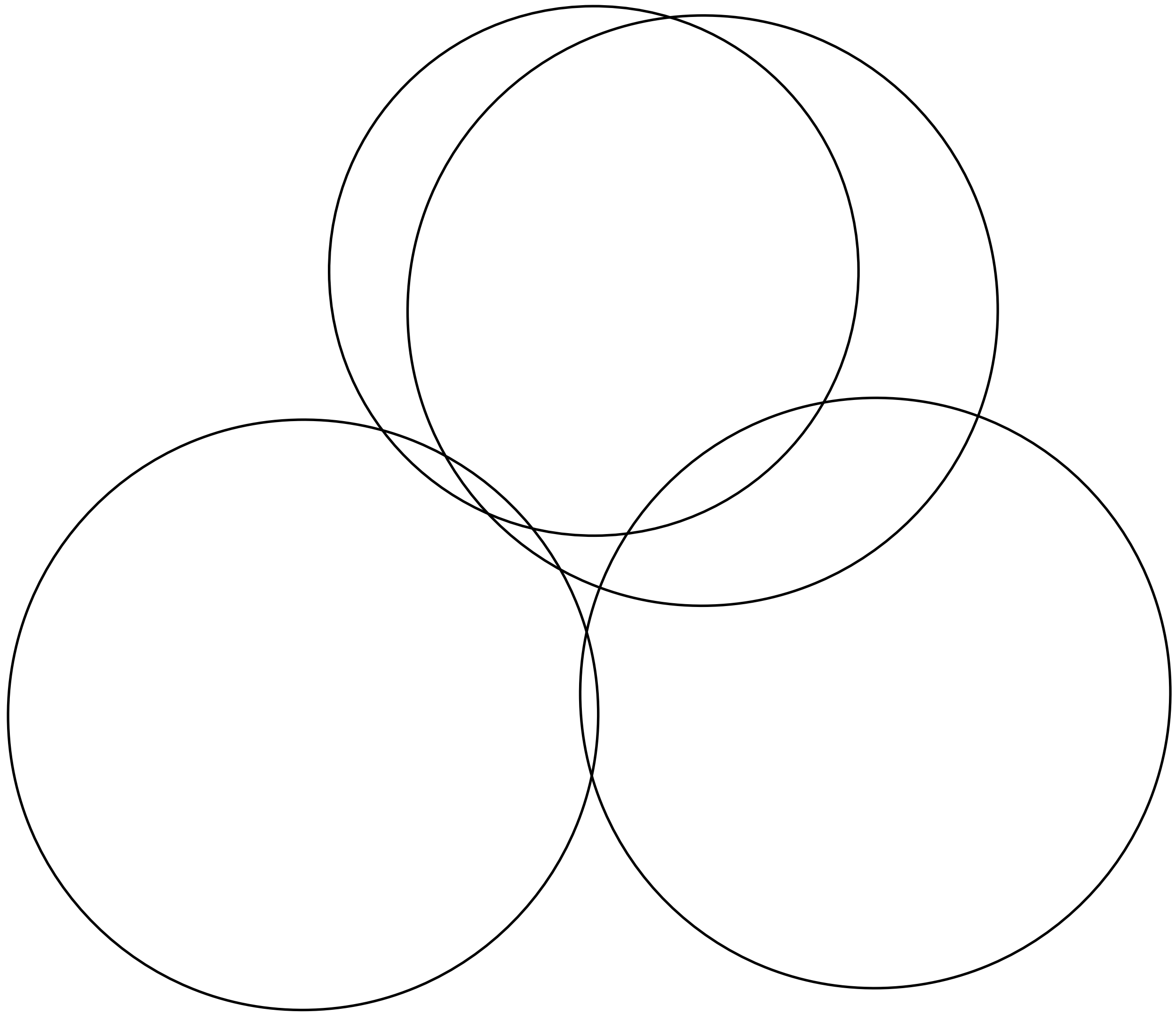
11,11,14,14 11

4*



11,12,14,14 * 7

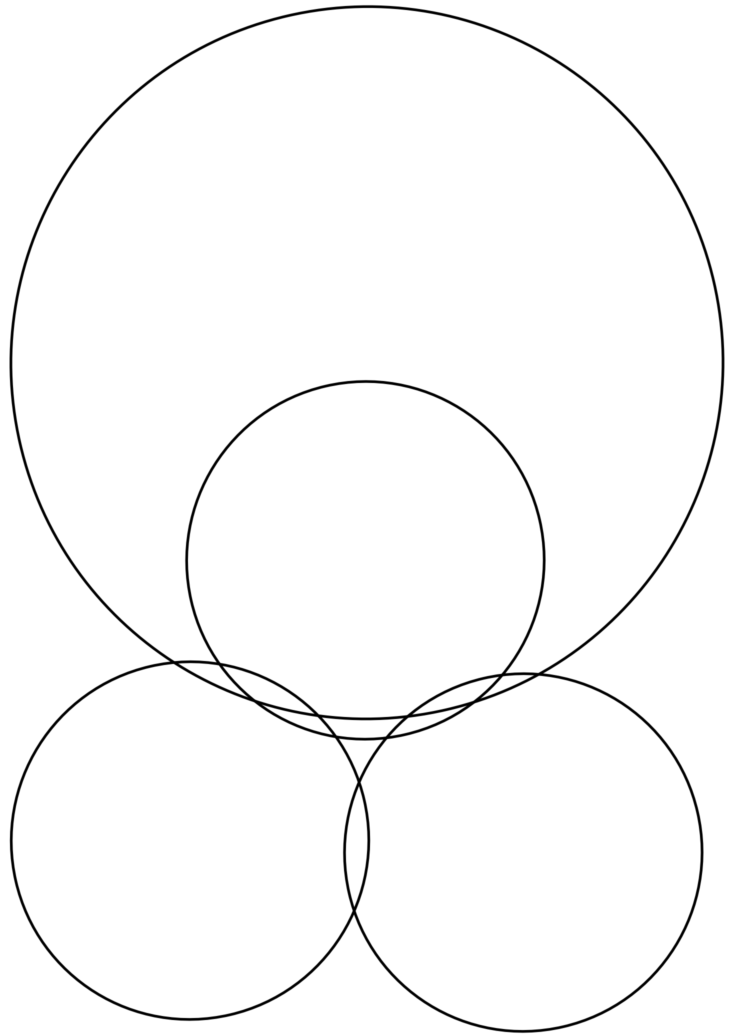
7*



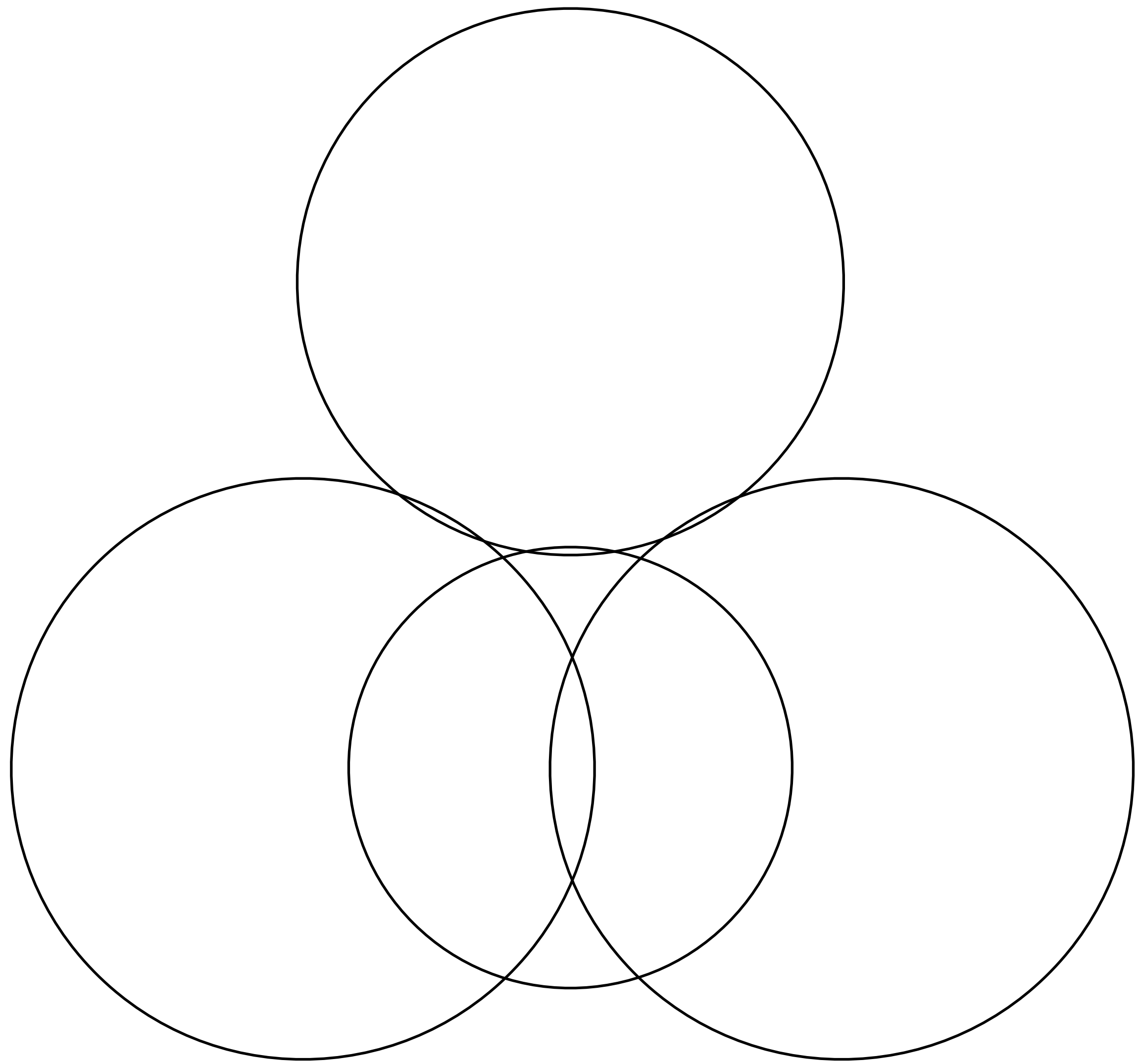
11,13,14,14*

3

3*

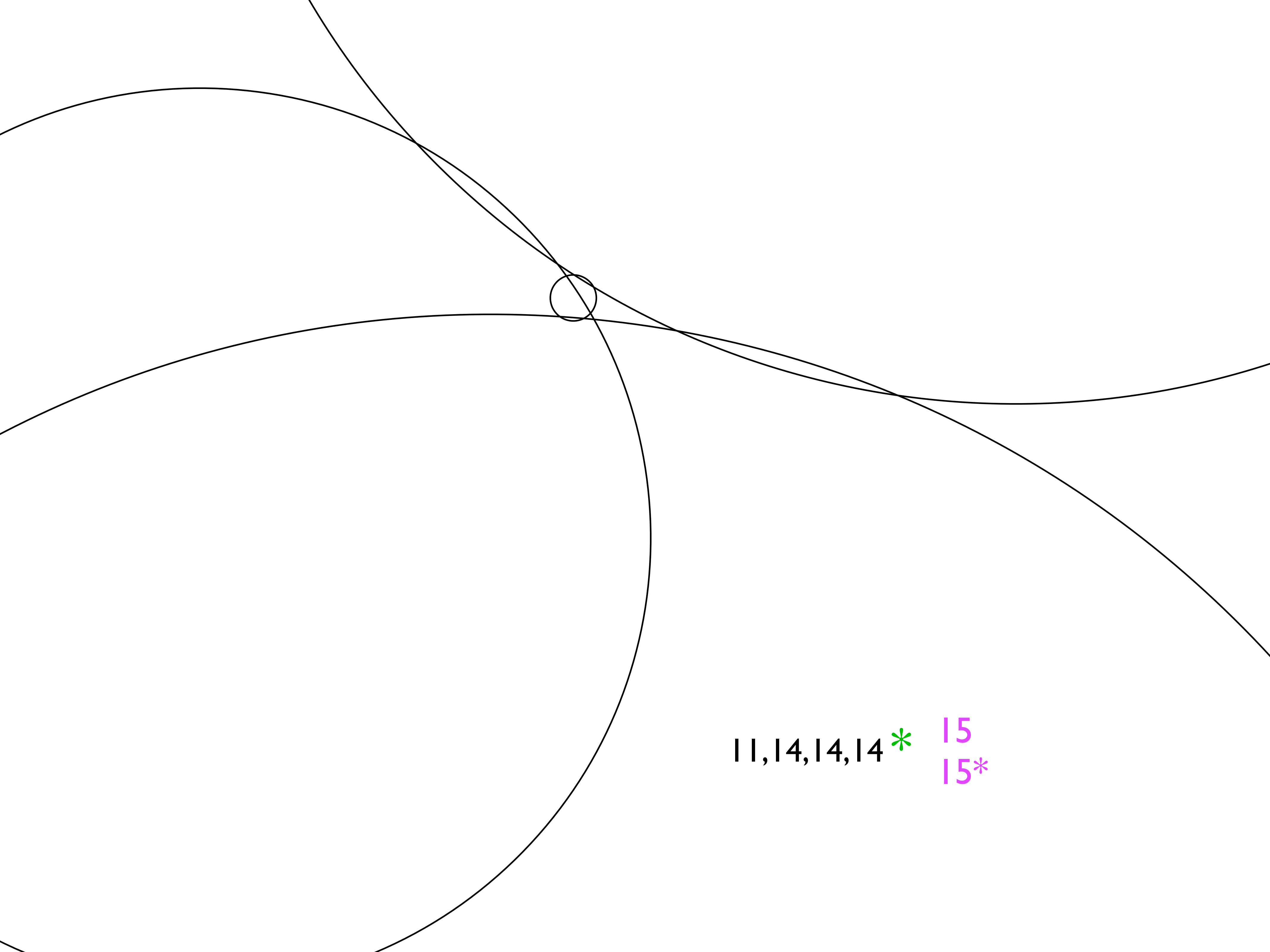


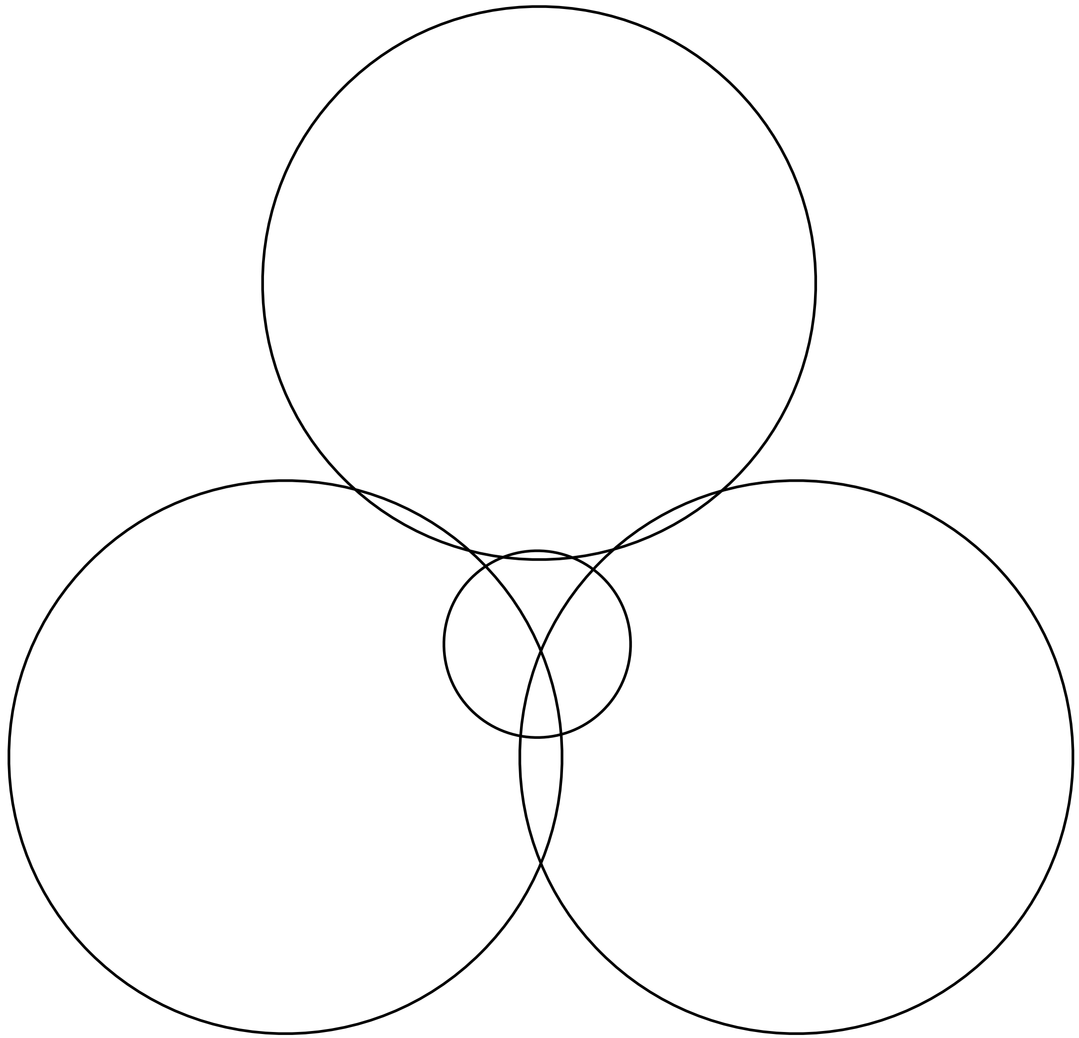
13,13,14,14



11,14,14,14

11
4*

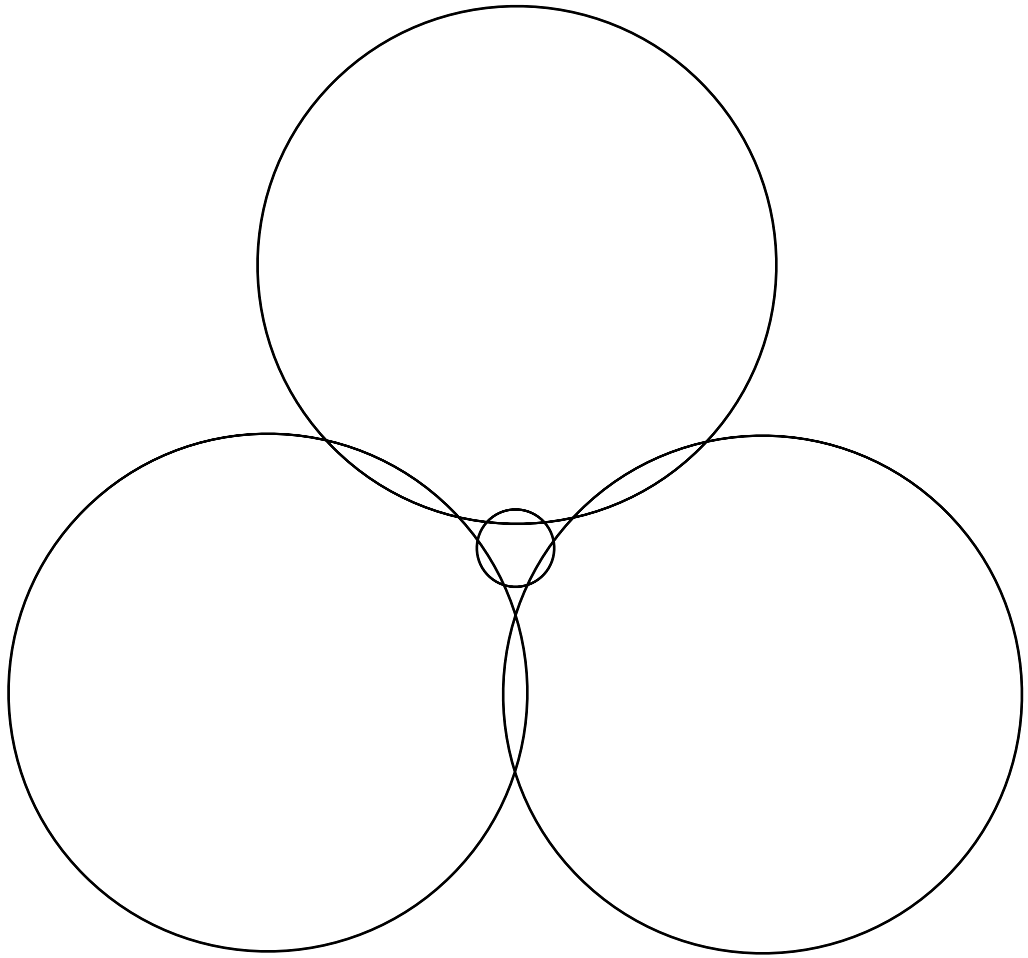

11,14,14,14*
15
15*



13,14,14,14

5

2*



14,14,14,14 19

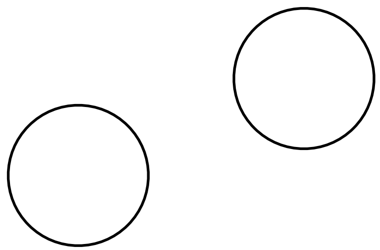
4*

That’s a total of 168 for n=4.

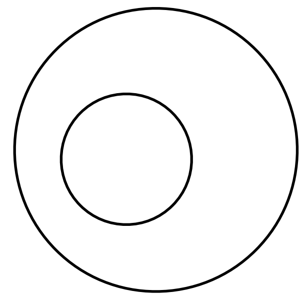
They were generated by hand according to a careful procedure that makes sure not to miss any.
Later I’ll show how to automate the process.

First, here are some possible extensions.
(I’ll call what came before *regular* configurations.)

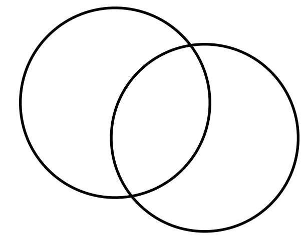
If we allow circles to be tangent to one another, a pair of circles enjoys two new possible relationships:



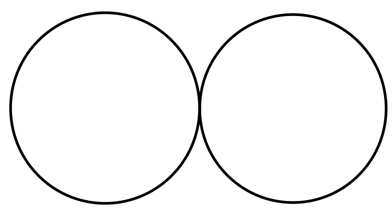
2-1



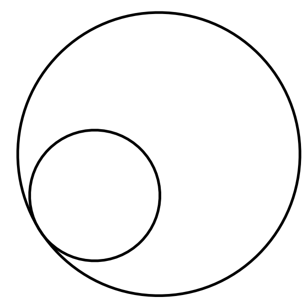
2-2



2-3

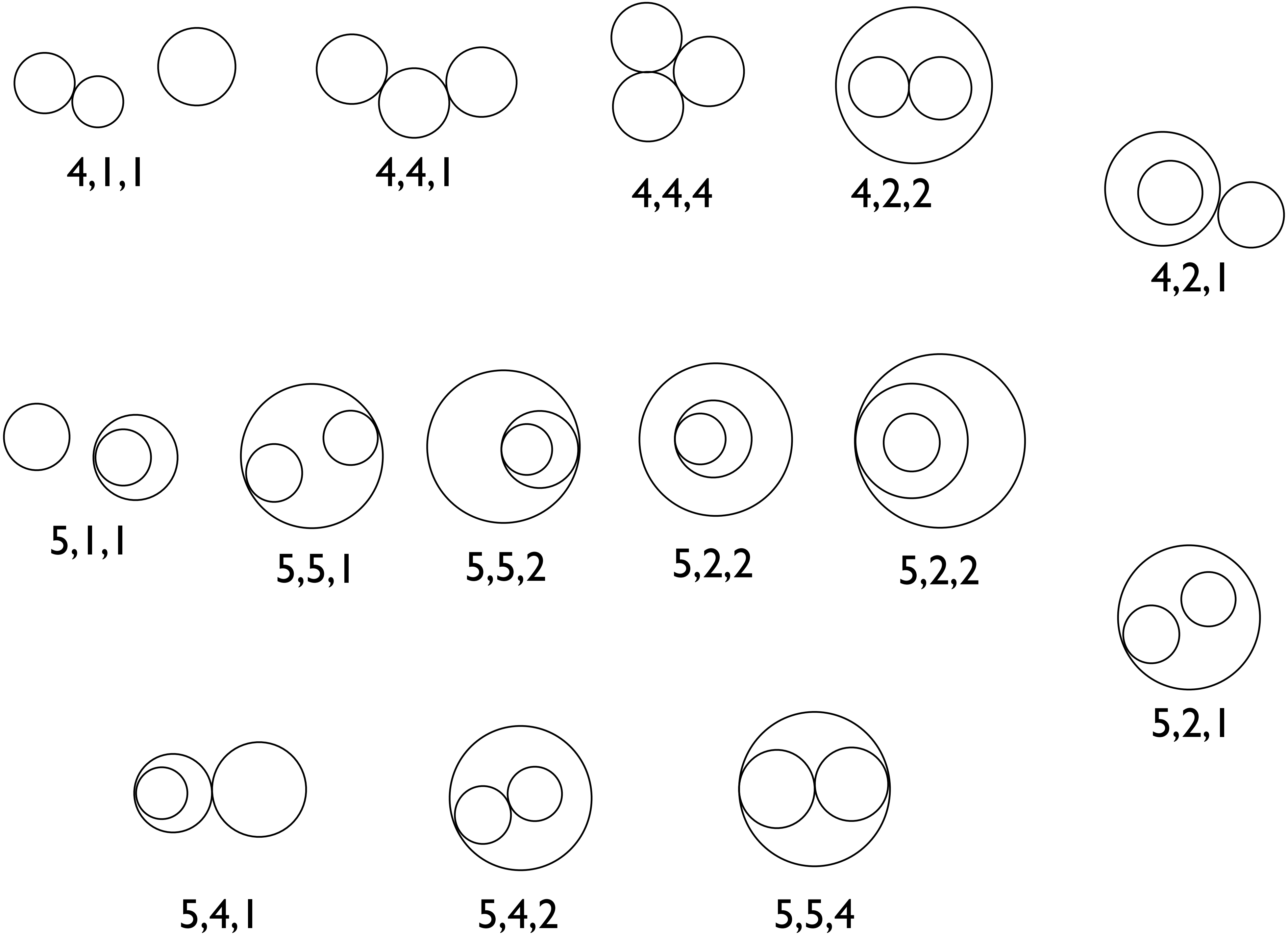


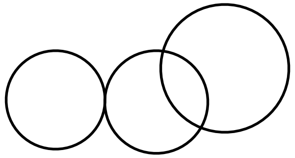
2-4



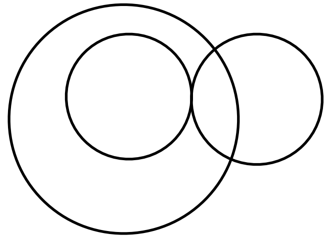
2-5

Next are shown all the 3-configurations that include tangent relationships, identified with their 2-decks:

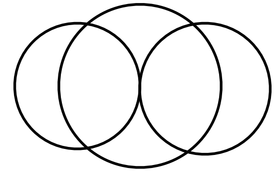

4,1,1
4,4,1
4,4,4
4,2,2
4,2,1
5,1,1
5,5,1
5,5,2
5,2,2
5,2,2
5,2,1
5,4,1
5,4,2
5,5,4



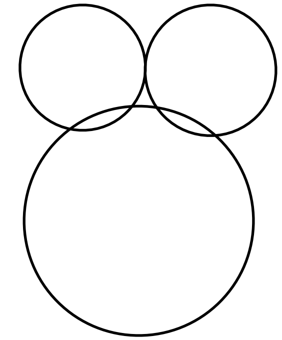
4,3,1



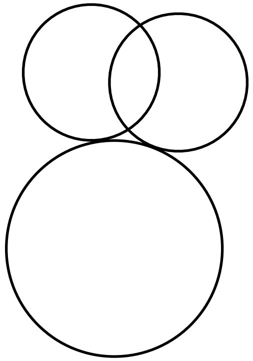
4,3,2



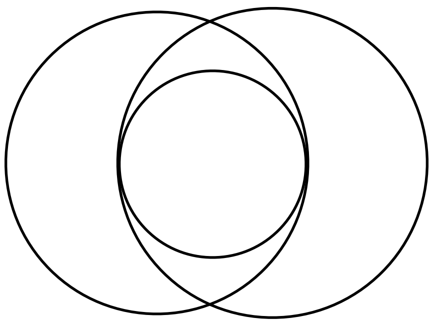
4,3,3



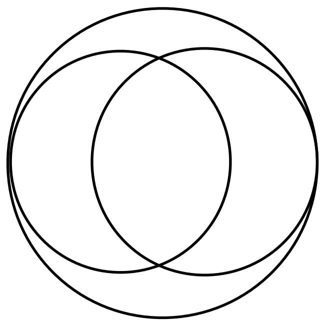
4,3,3



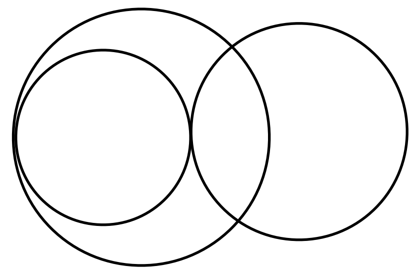
4,4,3



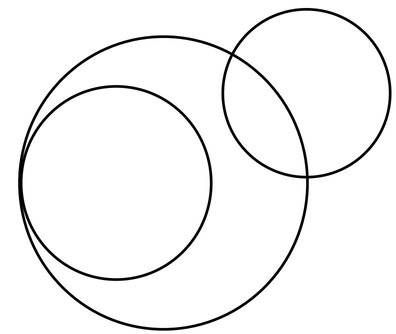
5,5,3



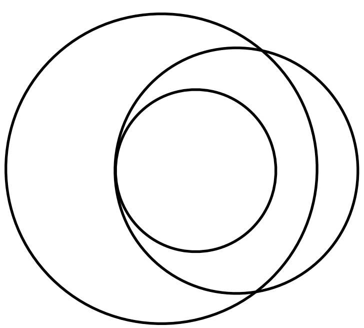
5,5,3



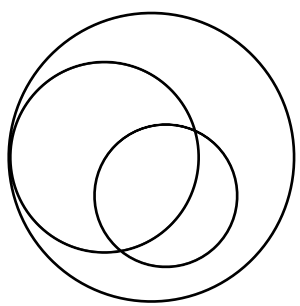
5,4,3



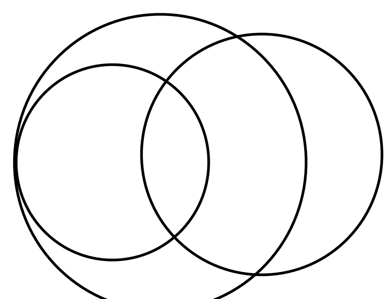
5,3,1



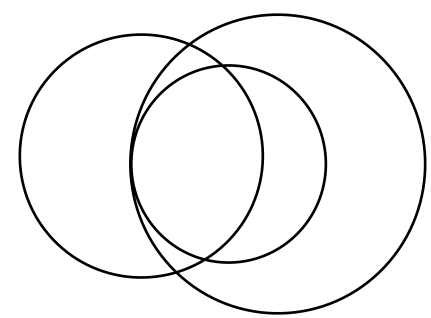
5,3,2



5,3,2



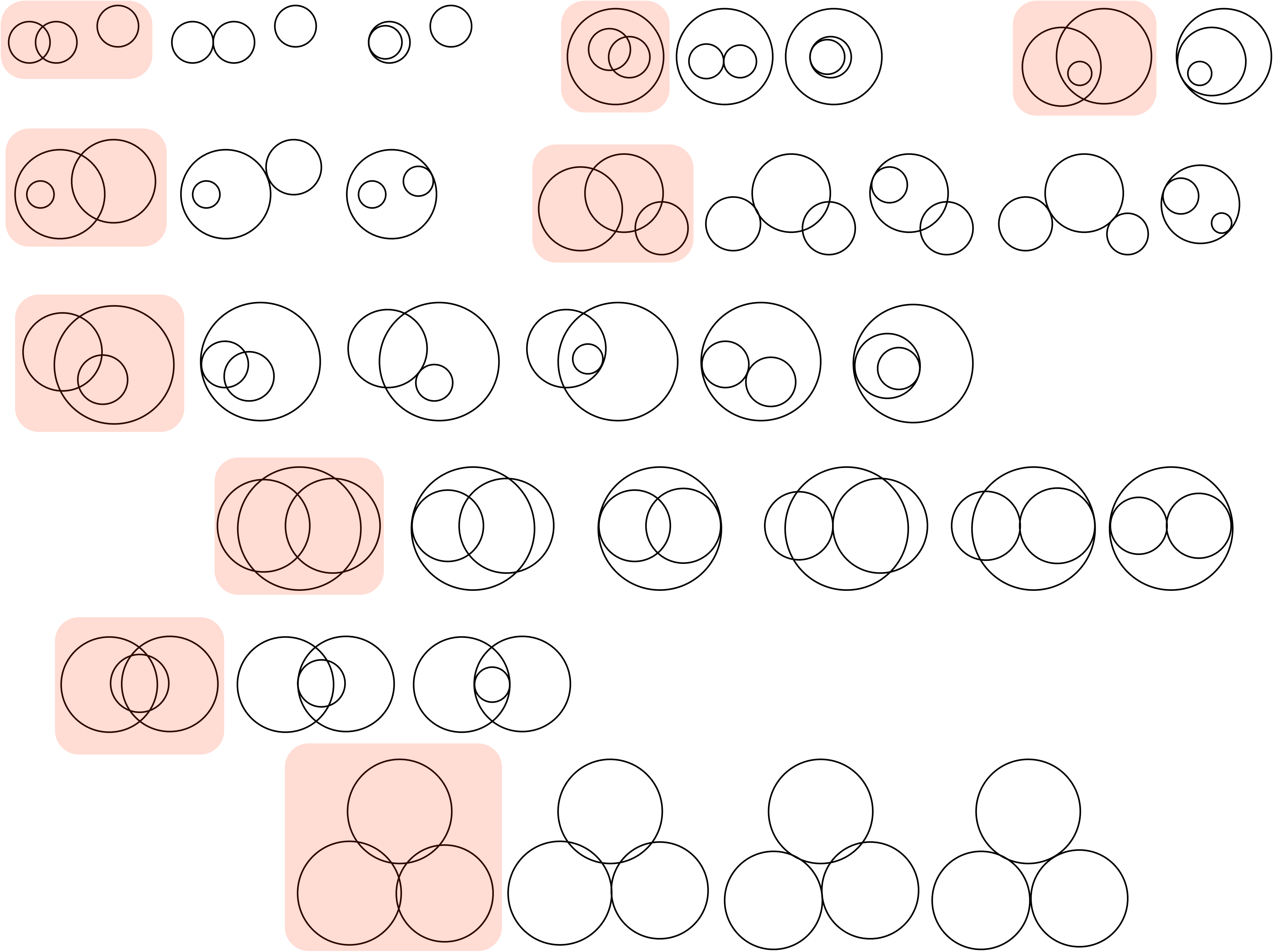
5,3,3



5,3,3

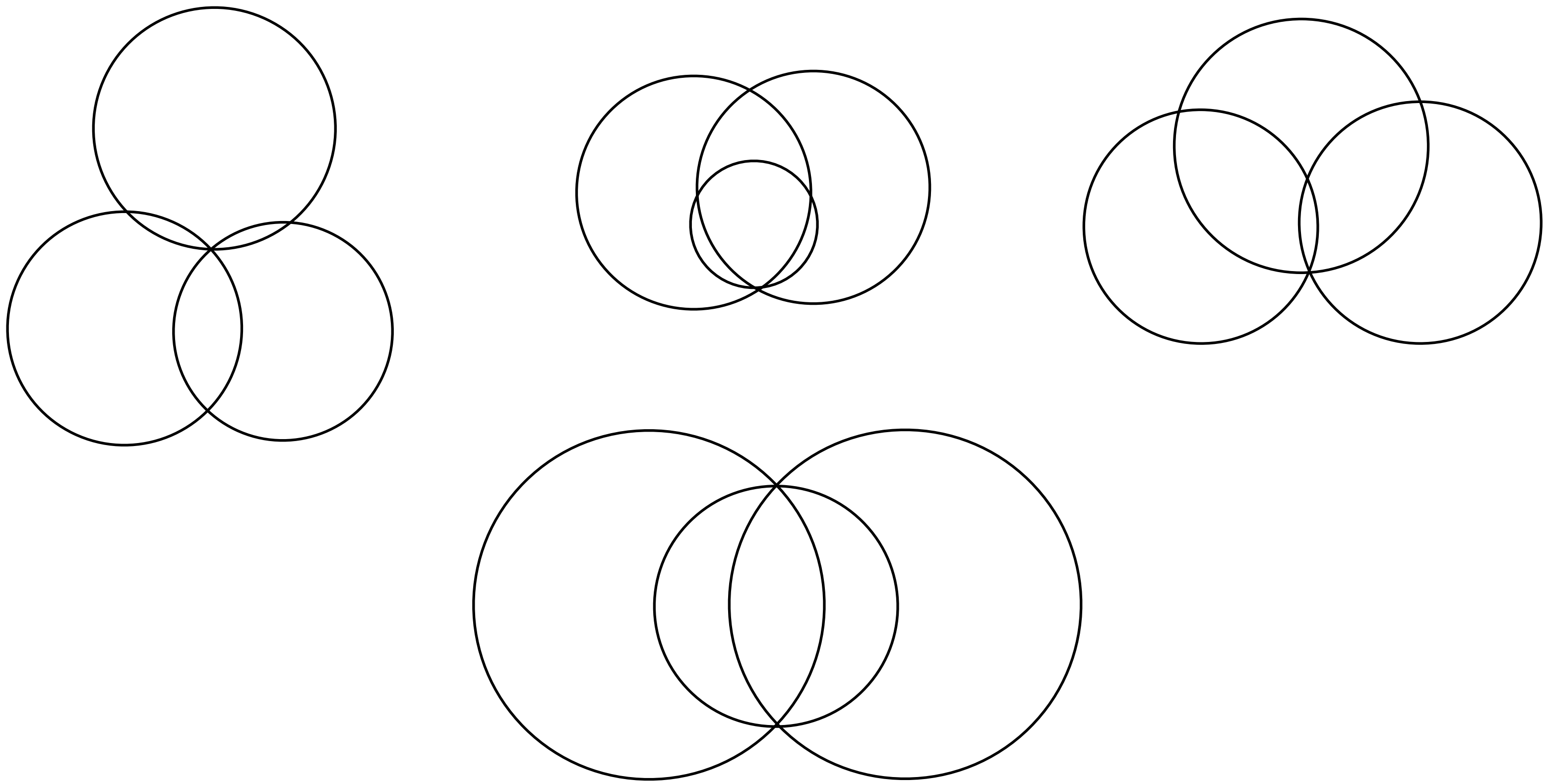
The pink numbers in the slides above for n=4 count the number of additional configurations it is possible to generate from each regular configuration by shrinking intersections to create tangencies. I'll show in the next slide how that works for generating the tangent configurations for n=3.

There turn out to be 1138 4-configurations if we allow tangent circles.

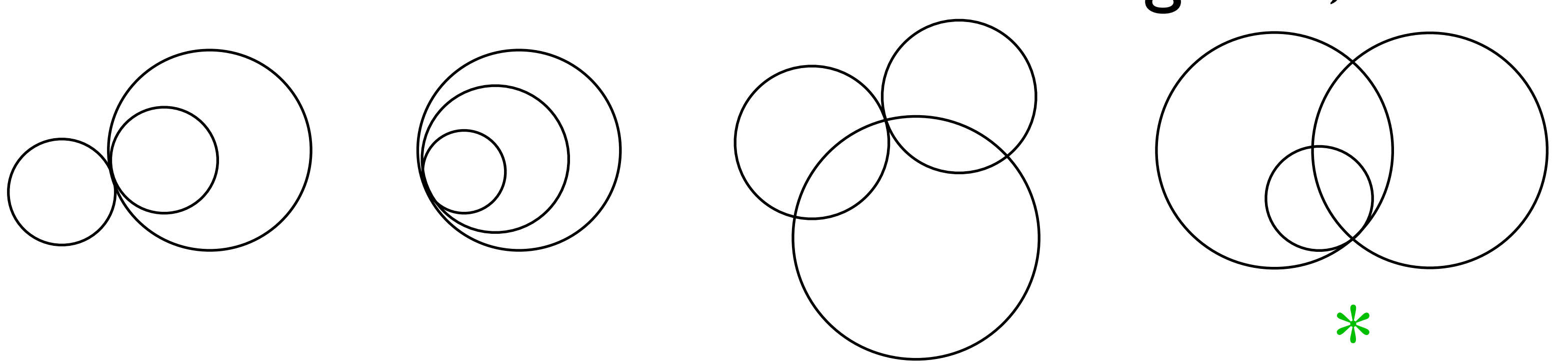


The blue numbers in the slides above for n=4 count the number of configs you get if you allow tangent relationships, but no interlocking relationships.

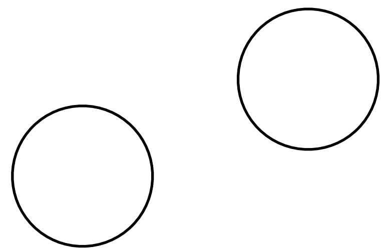
If we allow *coincidences*, i.e. where three circles meet at a single point, we have some new 3-configurations:



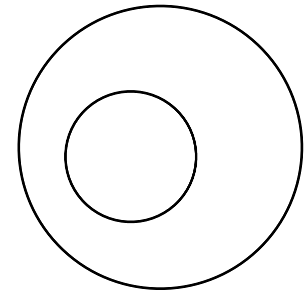
And if we allow coincidences *and* tangents, add these:



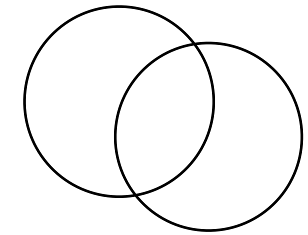
If we allow two circles to be completely congruent we have a 6th relationship between two circles:



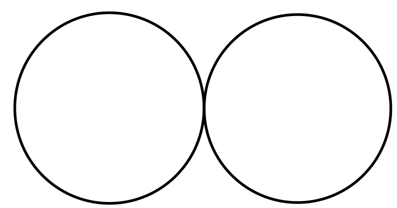
2-1



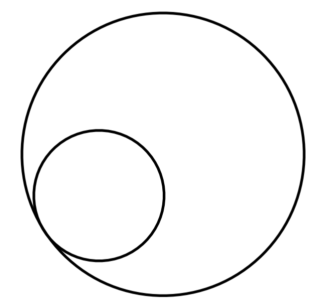
2-2



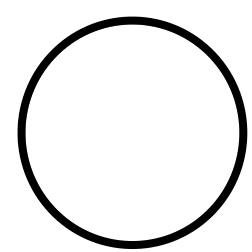
2-3



2-4

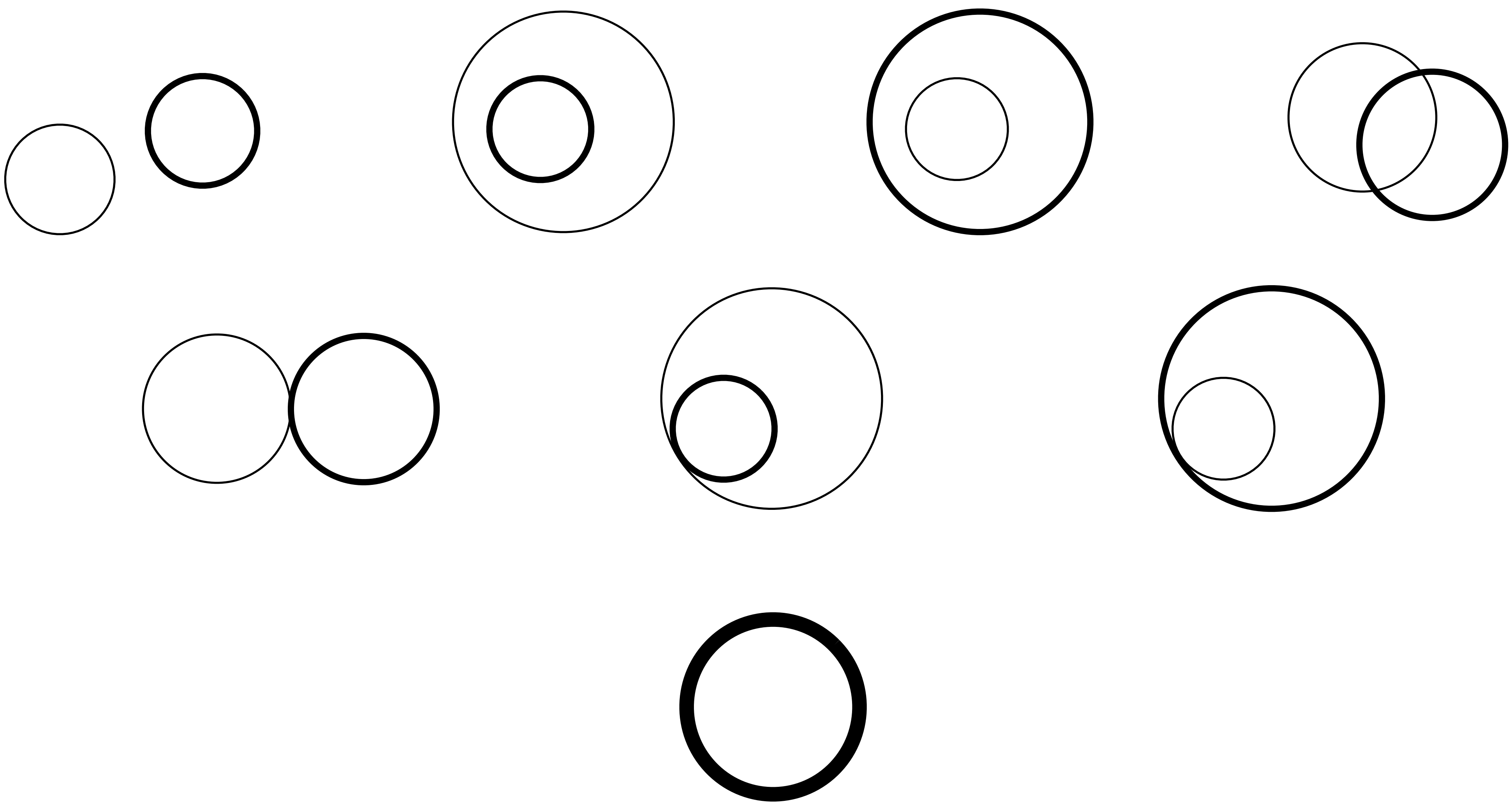


2-5



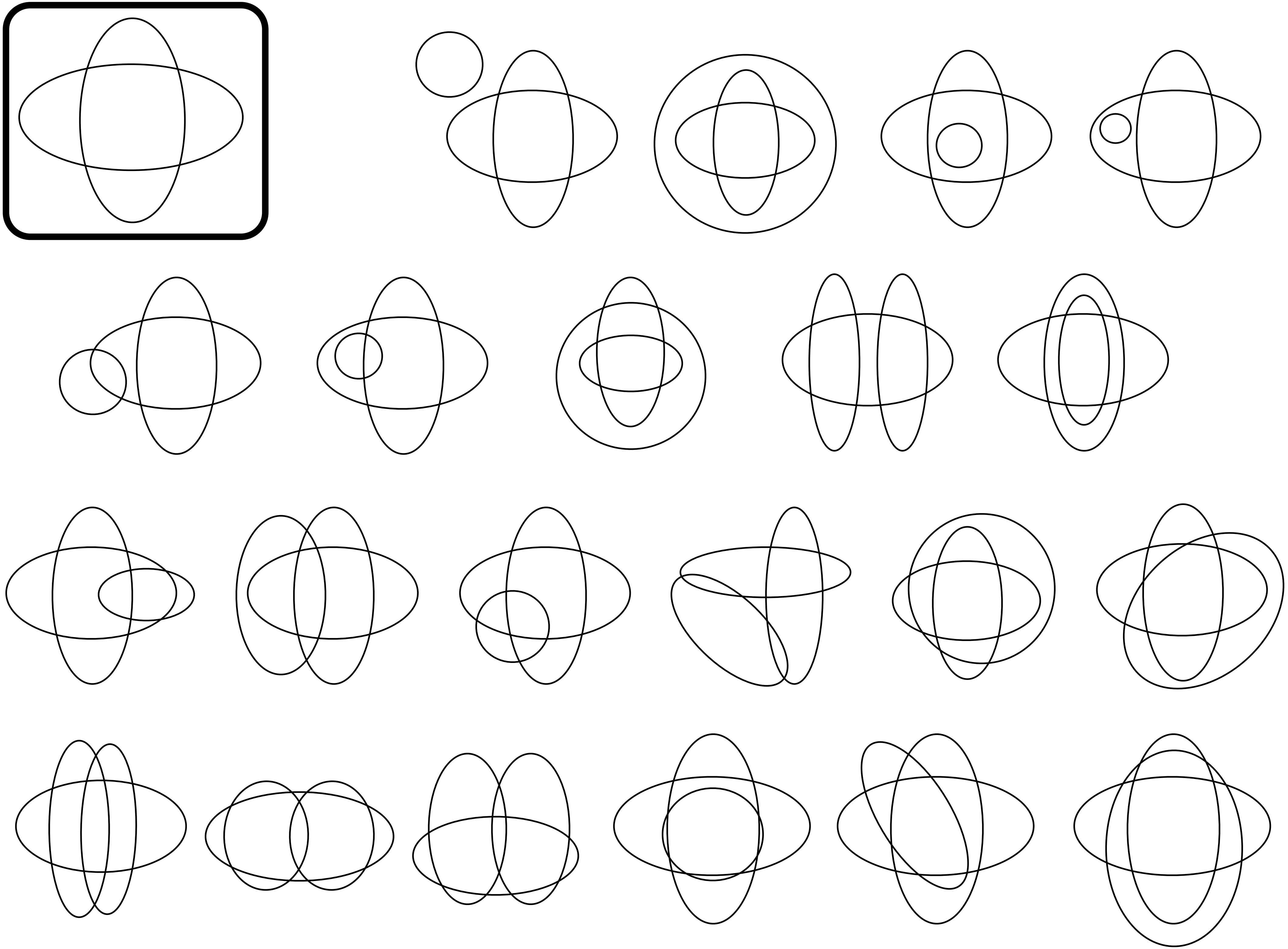
2-6

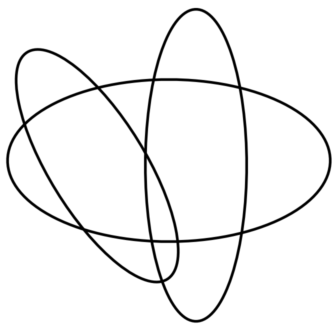
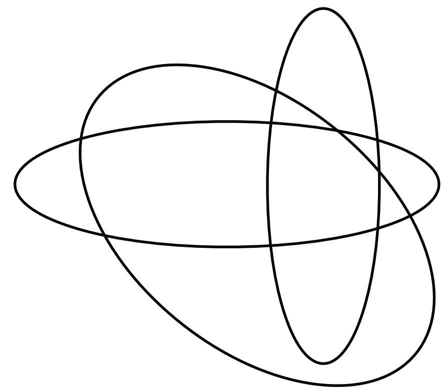
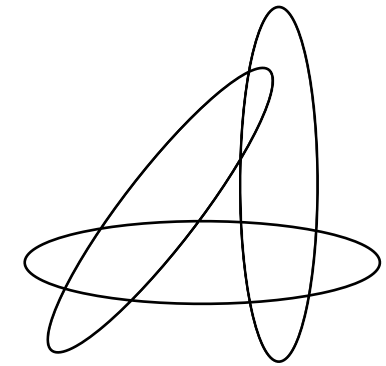
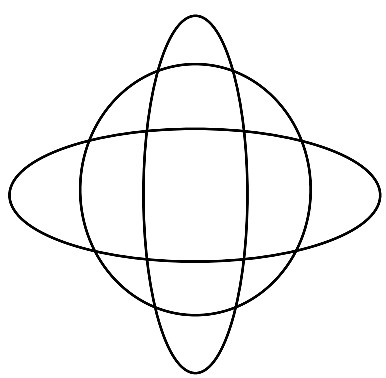
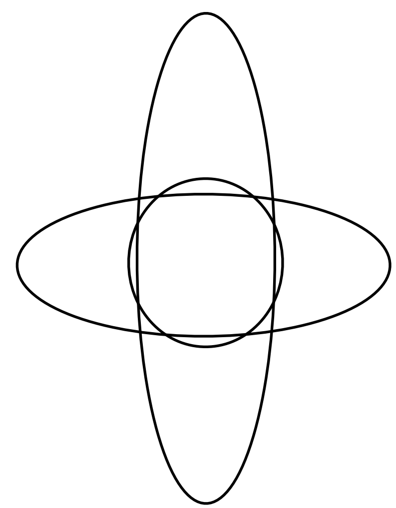
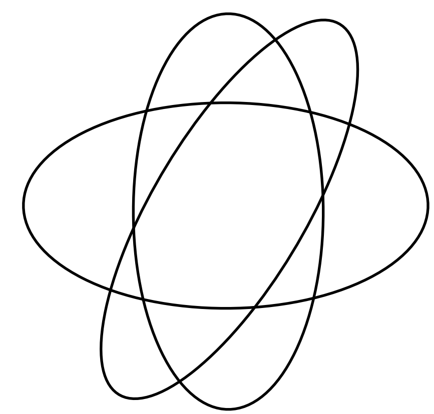
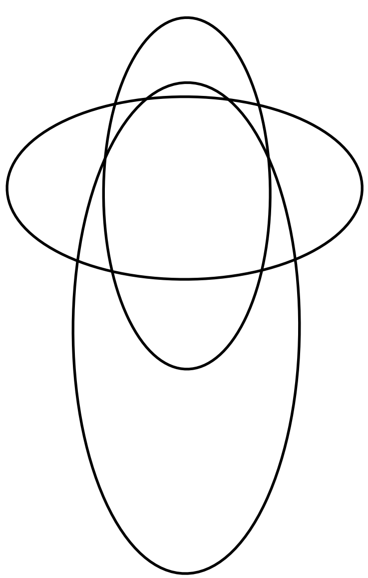
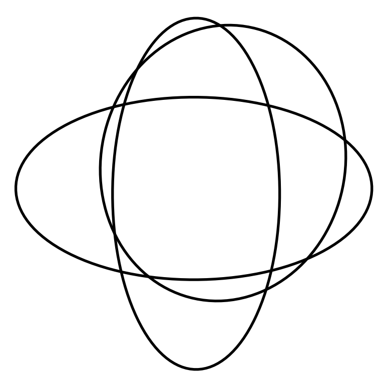
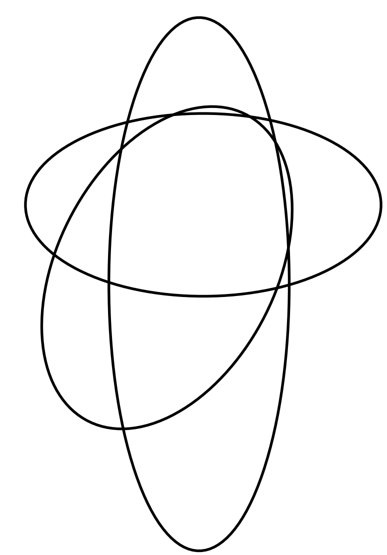
The new 3-configurations are as follows:



We could do ellipses, too...

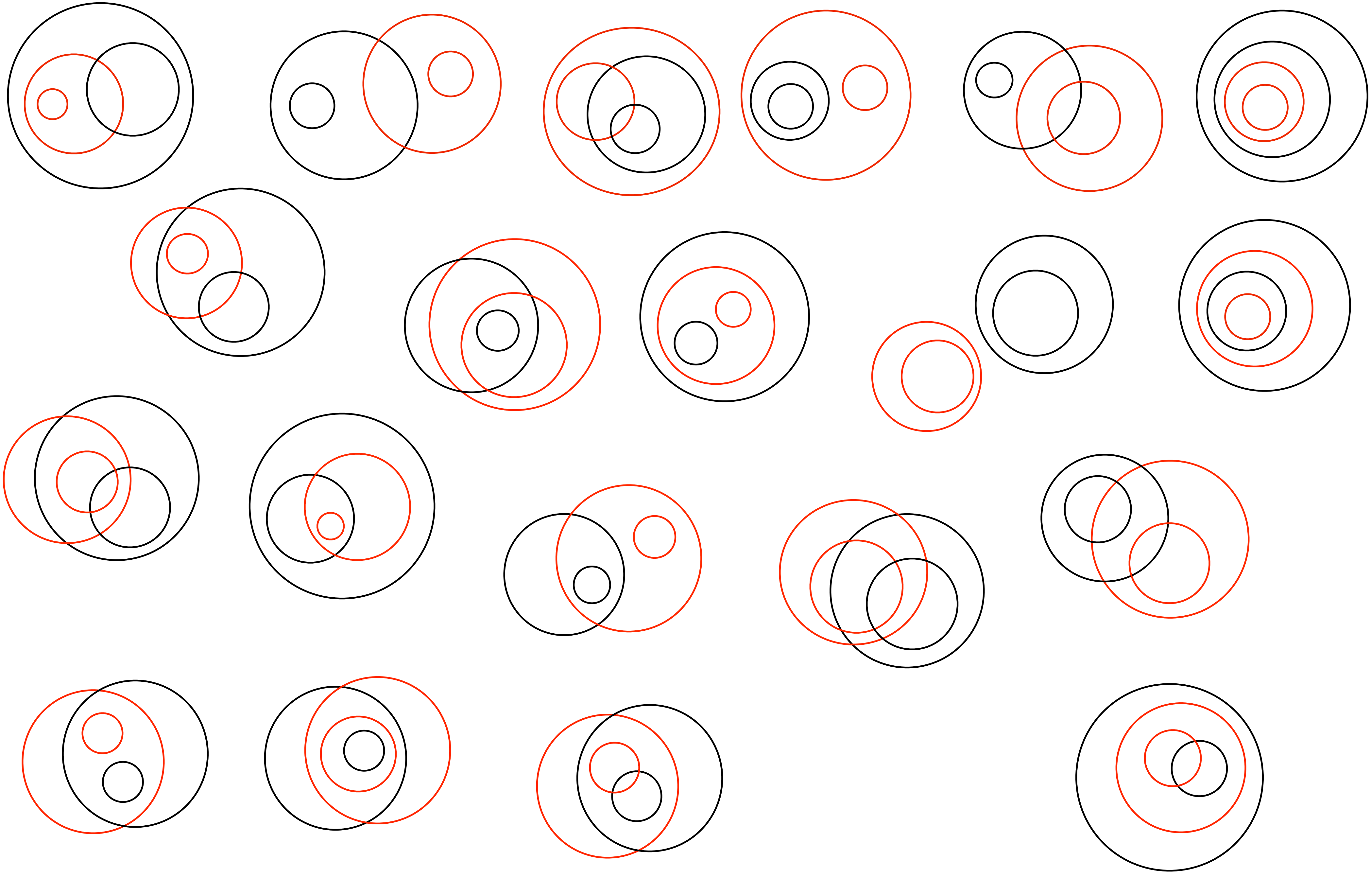
This is harder and the following are not completely worked out, but they're pretty and here they are (n=3) just for fun (the cases that do not require ellipses are not shown):



Or we could do rings (annuli) instead of circles.
This is harder.

For just two rings (not necessarily concentric)
we get these 20 configs:



Each of these could be limited to:

- only concentric annuli
- only congruent ellipses/annuli
- only same-sized ellipses/annuli/circles
- only congruent **and** same-sized ellipses/annuli/circles

etc.