1 Tonehole Modeling

Woodwind toneholes not only provide a means for bore length variation but also control sound radiation behavior according to their radius, height, and if present, keypad location. The influence of open and closed toneholes on wave propagation within a musical instrument bore greatly complicates the ideal behaviors previously discussed.

1.1 A Two-Port Tonehole Model

- The fundamental acoustic properties of toneholes have been extensively studied and reported by Keefe [1981, 1990a], Dubos et al. [1999], Dalmont et al. [2002], Nederveen et al. [1998], Lefebvre and Scavone [2012].
- The model described by Keefe [1990a] is an accurate representation for a tonehole unit, assuming adjacent tonehole interactions are negligible.
- In this description, acoustic variables at the tonehole junction are related by a transfer matrix of series and shunt impedance parameters.
- Keefe’s original derivation of the tonehole parameters was based on a symmetric T section, as shown in Fig. 1 [Keefe, 1981].

![Figure 1: T section transmission-line representation of the tonehole.](image)

- The series impedance terms, \( Z_a \), result from an analysis of anti-symmetric pressure distribution, or a pressure node, at the tonehole junction. In this case, volume flow is symmetric and equal across the junction.
- The shunt impedance term, \( Z_s \), results from an analysis of symmetric pressure distribution, or a pressure anti-node, at the tonehole, for which pressure is symmetric and equal across the junction.
- The transfer matrix that results under this analysis is given by

\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{Z_a}{2} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
Z_s^{-1} & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{Z_a}{2} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
P_2 \\
U_2
\end{bmatrix}
\]

(1)

\[
= \begin{bmatrix}
1 + \frac{Z_s Z_a}{2Z_s} & Z_s \left(1 + \frac{Z_a}{2Z_s}\right) \\
Z_s^{-1} & 1 + \frac{Z_a}{2Z_s}
\end{bmatrix}
\begin{bmatrix}
P_2 \\
U_2
\end{bmatrix}
\]

(2)

obtained by cascading the three matrices that correspond to the three impedance terms.
Based on the approximation that \(|Z_a/Z_s| \ll 1\), Eq. (2) can be reduced to the form
\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
1 & Z_a/Z_s^1 \\
Z_s^{-1} & 1
\end{bmatrix} \begin{bmatrix}
P_2 \\
U_2
\end{bmatrix},
\]
which is the basic tonehole unit cell given by Keefe for transfer-matrix calculations.

The values of \(Z_a\) and \(Z_s\) vary according to whether the tonehole is open (o) or closed (c) as
\[
\begin{align*}
Z_s^{(o)} &= Z_c(a/b)^2(jkt_e + \xi_e), \\
Z_s^{(c)} &= -jZ_c(a/b)^2\cot(kt), \\
Z_a^{(o)} &= -jZ_c(a/b)^2k_d^{(o)}, \\
Z_a^{(c)} &= -jZ_c(a/b)^2k_d^{(c)}.
\end{align*}
\]

Definitions and descriptions of the various parameters in Eqs. (4) – (7) can be found in Keefe [1990a].

To render these relationships in the digital waveguide domain, it is necessary to transform the plane-wave physical variables of pressure and volume velocity to traveling-wave variables as
\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
P_1^+ + P_1^- \\
Z_c^{-1}(P_1^+ - P_1^-)
\end{bmatrix},
\]
where \(Z_c\) is the characteristic impedance of the cylindrical bore, which is equal on both sides of the tonehole.

Waveguide pressure variables on both sides of the tonehole are then related by
\[
\begin{bmatrix}
P_1^- \\
P_2^+
\end{bmatrix} = \begin{bmatrix}
R^- & T^- \\
T^+ & R^+
\end{bmatrix} \begin{bmatrix}
P_1^+ \\
P_2^-
\end{bmatrix},
\]
where
\[
\begin{align*}
R^- &= R^+ \approx \frac{Z_aZ_s - Z_c^2}{Z_aZ_s + 2Z_cZ_s + Z_c^2}, \\
T^- &= T^+ \approx \frac{2Z_cZ_s}{Z_aZ_s + 2Z_cZ_s + Z_c^2},
\end{align*}
\]
calculated using Eqs. (2) and (8) and then making appropriate simplifications for \(|Z_a/Z_s| \ll 1\).

Figure 2 depicts the waveguide tonehole two-port scattering junction in terms of these reflectances and transmittances. This structure is analogous to the four-multiply Kelly-Lochbaum scattering junction [Kelly and Lochbaum, 1962].

For the implementation of the reflectances and transmittances given by Eqs. (10) – (11) in the digital waveguide structure of Fig. 2, it is necessary to convert the continuous-time filter responses to appropriate discrete-time representations.
• Results of this approach are shown in Figure 3 and are compared with reproduced results using the technique of Keefe [Keefe, 1990a] for a simple flute air column with six toneholes.

• The implementation of a sequence of toneholes in this way is diagrammed in Figure 4.

1.2 A Three-Port Tonehole Model

• For the purpose of real-time modeling, the two-port implementation has a particular disadvantage: the two lumped characterizations of the tonehole as either closed or open cannot be efficiently unified into a single tonehole model.

• It is preferable to have one model with adjustable parameters to simulate the various states of the tonehole, from closed to open and all states in between.
To this end, it is best to consider a distributed model of the tonehole, such that “fixed” portions of the tonehole structure are separated from the “variable” component.

The junction of the tonehole branch with the main air column of the instrument can be modeled in the DW domain using a three-port scattering junction, as described in Scavone [1997].

This method inherently models only the shunt impedance term of the Keefe tonehole characterization, however, the negative length correction terms implied by the series impedances can be approximated by adjusting the delay line lengths on either side of the three-port scattering junction.

The other “fixed” portion of the tonehole is the short branch segment itself, which is modeled in the DW domain by appropriately sized delay lines.

This leaves only the characterization of the open/closed tonehole end.

A simple inertance model of the open hole end offers the most computationally efficient solution. The impedance of the open end is then given by

$$Z_{e}^{(o)}(s) = \frac{\rho t}{A_e} s,$$

where $\rho$ is the density of air, $A_e$ is the cross-sectional area of the end hole, $t$ is the effective length of the opening ($\approx A_e^{1/2}$), and $s$ is the Laplace transform frequency variable.

The open-end reflectance is

$$R_{e}^{(o)}(s) \triangleq \frac{P_e^{-}(s)}{P_e^{+}(s)} = \frac{Z_{e}^{(o)}(s) - Z_{cb}}{Z_{e}^{(o)}(s) + Z_{cb}} = \frac{tA_b s - cA_e}{tA_b s + cA_e},$$

where $Z_{cb} = \rho c/A_b$ is the characteristic impedance of the tonehole branch waveguide, $A_b$ is the cross-sectional area of the branch and $c$ is the speed of sound.

An appropriate discrete-time filter implementation for $R_{e}^{(o)}$ can be obtained using the conformal bilinear transform from the $s$-plane to the $z$-plane [Oppenheim and Schafer, 1989, pp. 415-430], with the result

$$R_{e}^{(o)}(z) = \frac{a - z^{-1}}{1 - az^{-1}},$$

where

$$a = \frac{tA_b \alpha - cA_e}{tA_b \alpha + cA_e}$$

and $\alpha$ is the bilinear transform constant that controls frequency warping.

The discrete-time reflectance $R_{e}^{(o)}(z)$ is a first-order allpass filter, which is consistent with reflection from a “masslike” impedance.

It is possible to simulate the closing of the tonehole end by taking the end hole radius (or $A_e$) smoothly to zero.

In the above implementation, this is accomplished simply by varying the allpass coefficient between its fully open value and a value nearly equal to one.

With $a \approx 1$, the reflectance phase delay is nearly zero for all frequencies, which corresponds well to pressure reflection at a rigid termination.

A complete implementation scheme is diagrammed in Figure 5.

Figure 6 shows the reflection functions obtained using this model in comparison to the Keefe transmission-line results.

This efficient model of the tonehole produces results very much in accord with the more rigorous model.

A more accurate model of the tonehole branch end, which is not pursued here, would include a frequency-dependent resistance term and require the variation of three first-order filter coefficients.
1.3 Register Hole Models

- Woodwind register holes are designed to discourage oscillations based on the fundamental air column mode and thus to indirectly force a vibratory regime based on higher, more stable resonance frequencies.

- A register vent functions both as an acoustic inertance and an acoustic resistance [Benade, 1976]. It is ideally placed about one-third of the distance from the excitation mechanism of a cylindrical-bored instrument to its first open hole.

- Sound radiation from a register hole is negligible.

- The DW implementation of a register hole can proceed in a manner similar to that for the tonehole. The series impedance terms associated with toneholes are insignificant for register holes and can be neglected.

- Modeling the open register hole as an acoustic inertance in series with a constant resistance, its input impedance as seen from the main bore is given by

\[
Z_{rh}^{(o)}(s) = \frac{\rho t}{A_{rh}}s + \xi,
\]

where \(\rho\) is the density of air, \(t\) is the effective height, \(A_{rh}\) is the cross-sectional area of the hole, \(\xi\) is the acoustic resistance, and \(s\) is the Laplace transform frequency variable.

- Proceeding with a two-port DW implementation, the register hole is represented in matrix form by

\[
\begin{bmatrix}
P_1^- \\
P_2^-
\end{bmatrix} = \begin{bmatrix}
R^- & T^- \\
T^+ & R^+
\end{bmatrix} \begin{bmatrix}
P_1^+ \\
P_2^+
\end{bmatrix} = \frac{1}{Z_c + 2Z_{rh}} \begin{bmatrix}
-Z_c & 2Z_{rh} \\
2Z_{rh} & -Z_c
\end{bmatrix} \begin{bmatrix}
P_1^+ \\
P_2^+
\end{bmatrix},
\]

where the open register hole shunt impedance is given by \(Z_{rh}^{(o)}\) and \(Z_c\) is the characteristic impedance of the main air column.

- The reflectances and transmittances are equivalent at this junction for wave components traveling to the right or to the left. As \(T = 1 + R\), a one-filter form of the junction is possible.
Using the bilinear transform, an appropriate discrete-time implementation for $R_{rh}$ is given by

$$R_{rh}^-(z) = R_{rh}^+(z) = \frac{-c (1 + z^{-1})}{(\zeta + \alpha \psi) + (\zeta - \alpha \psi) z^{-1}},$$

(18)

where

$$\zeta = c + 2A_0 \xi / \rho \quad \text{and} \quad \psi = 2A_0 t / A_{rh},$$

(19)

$A_0$ is the cross-sectional area of the main air column, and $\alpha$ is the bilinear transform constant that controls frequency warping.

• Assuming the closed register hole has negligible effect in the acoustic model, simulated closure of the register hole in this implementation is achieved by ramping the reflectance filter gain to zero.

• This implementation is similar to that of Välimäki et al. [1993], though resistance effects were not accounted for in that study.

• As discussed by Benade [Benade, 1976, p. 459], a misplaced register hole will raise the frequency of the second air column mode by an amount proportional to its displacement from the ideal location (in either direction).

• Such behavior is well demonstrated when this register hole implementation is added to the real-time clarinet model.

## 2 Reed Valve Modeling

Wind instruments are “driven” by either pressure- or velocity-controlled valves. In this section we focus on the modeling of pressure-controlled valves, such as the clarinet reed and the brass player’s lips. While most of the focus here is on the reed valve, much of the analysis can be equally applied to the lip mechanism as well.
2.1 Pressure-Controlled Reeds

- The single-reed and mouthpiece arrangement of clarinets and saxophones (Fig. 7) acts as a pressure-controlled valve that allows energy into the instrument for the initialization and maintenance of oscillations in the downstream air column.

![Figure 7: A single-reed woodwind mouthpiece.](image)

- The inherent non-linear behavior of this mechanism, which is attributable to such aspects as the flow characteristic through the reed aperture and the reed’s displacement when it hits the mouthpiece facing, is most obviously demonstrated by the fact that a nearly static pressure applied at the upstream side of the system is converted into acoustic energy on the downstream side at a number of harmonically related frequencies.

- The flow and reed movement are controlled by the difference in pressures on the upstream and downstream sides of the reed channel, $p_\Delta = p_u - p_d$.

- The upstream pressure is typically assumed constant or slowly varying and tends to force the reed toward the mouthpiece lay.

- Negative pressure in the mouthpiece reinforces this action, pulling the reed toward the lay, while positive pressure in the mouthpiece, if acting alone, pushes the reed away from the mouthpiece lay.

- The single-reed valve is initially open but can be blown shut against the mouthpiece lay by an appropriate pressure difference $p_\Delta$.

- The movement of the reed controls the volume flow through the reed channel and into the mouthpiece.

2.2 Vibrations of the Reed

- Some of the earliest research on musical instrument reeds was conducted by Helmholz, who concluded that pipes excited by inwardly striking reeds “of light material which offers but little resistance” produce tones at frequencies corresponding to the resonant frequencies of the pipe, which are much lower than the natural frequency of the reed itself [Helmholtz, 1954, p. 390].

- The lowest resonance frequency of a typical clarinet reed falls approximately in the range 2-3 kHz, while normal playing frequencies for clarinets are below 1 kHz.

- A mass-spring system driven at a frequency well below resonance is said to be stiffness dominated and its displacement amplitude will approach $f/k$, where $k$ is the spring constant and $f$ is the applied force.

- Thus, a common simplification for woodwind instruments has been to neglect the effect of the mass altogether, which is equivalent to assuming an infinite reed resonance frequency, and to model the reed system as a memory-less system as depicted in Fig. 8 [Backus, 1963, Nederveen, 1969, McIntyre et al., 1983].

- Observations by Backus [1961] of reed motion and mouthpiece pressure in an artificially blown clarinet appear to confirm that the reed is primarily stiffness controlled.
• Assuming the force on the reed is equal to \( A_r \cdot p_\Delta \), where \( A_r \) is the effective surface area of the reed exposed to \( p_\Delta \), the reed tip opening with respect to its equilibrium position \( H \) is given by Hooke’s law as

\[
x = H - \frac{A_r \cdot p_\Delta}{k}.
\]

(20)

• In this case, the motion of the reed is exactly “in-phase” with the pressure \((p_\Delta)\) acting on it.

2.3 Flow Through the Reed Orifice

• The flow through the reed orifice is typically determined using the Bernoulli equation for steady, laminar flow of an incompressible fluid [Backus, 1963, Worman, 1971, Wilson and Beavers, 1974, Thompson, 1979, Saneyoshi et al., 1987, Fletcher and Rossing, 1991].

• Relating pressure \((p)\) and velocity \((v)\) at any two points of height \(y\) within a continuous tube of flow, the Bernoulli equation is given by

\[
p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2,
\]

(21)

where \(\rho\) is fluid density and \(g\) is the acceleration of gravity. This expression is based on continuity of volume flow and conservation of energy.

• For the reed geometry as approximated in Fig. [9] the upstream cavity is viewed as a large tank of constant or slowly varying pressure \(p_u\) and essentially zero volume flow \(u_u\).

• Application of the Bernoulli equation, assuming no appreciable change in height, leads to an expression for the flow through the reed orifice of the form

\[
u = w x \left( \frac{2 |p_\Delta|}{\rho} \right)^{1/2} \text{sgn}(p_\Delta),
\]

(22)
where \( w \) and \( x \) are the width and height of the reed channel, respectively, and the sign function is

\[
\text{sgn}(p_\Delta) = \begin{cases} 
+1 & \text{if } p_\Delta \geq 0 \\
-1 & \text{otherwise.}
\end{cases}
\]

- An example flow curve determined using Eqs. (22) is shown in Fig. 10.

![Flow characteristic through the reed orifice.](image)

- Although the flow expression of Eq. (22) is derived for steady flow, its use is generally assumed to be valid under oscillating conditions as well. The validity of this assumption has been questioned by results of da Silva et al. [2007].

- The acoustic resistance to flow presented by the orifice is inversely proportional to the slope of the flow curves in Fig. 10 [Fletcher and Rossing, 1991]. For low values of \( p_\Delta \), the resistance is given by a small positive value. At very high pressure differences, the reed is blown closed against the mouthpiece facing and all flow ceases.

- The threshold blowing pressure, which occurs at the flow peak, can be determined from Eqs. (22) and (20) and is roughly equal to 1/3 the closure pressure. Between the threshold and closure pressures, the flow resistance is negative and the reed mechanism functions as an acoustic generator.

### 2.4 The Reed-Spring Solution

- Under the previous assumption that the reed can be adequately modeled as a mechanical spring, the motion of which is in phase with the pressure \( (p_\Delta) \) acting across it, Eq. (22) can be rewritten using Eq. (20) as

\[
u = w \left( H - \frac{A_r \cdot p_\Delta}{k} \right) \left( \frac{2p_\Delta}{\rho} \right)^{1/2} \text{sgn}(p_\Delta)
\]

\[
u = wH \left( 1 - \frac{p_\Delta}{p_c} \right) \left( \frac{2p_\Delta}{\rho} \right)^{1/2} \text{sgn}(p_\Delta), \tag{23}
\]

where \( p_c = kH/A_r = \mu_r H \omega^2 \) is the pressure necessary to push the reed against the mouthpiece facing and completely close the reed channel.
• The flow through the reed orifice is assumed to separate and form a free jet at the air column entrance. In the ensuing region of turbulence, the pressure is assumed equivalent to the reed channel pressure and conservation of volume flow to hold.

• In this case, \( u = u_d \) and we can use Eq. (23) and the relation

\[
    u_d(t) = u_d^+(t) + u_d^-(t) = Z_c^{-1} [p_d^+(t) - p_d^-(t)],
\]

(24)
to find a set of solutions for various values of \( p_\Delta \).

• The approach of McIntyre et al. [1983] is to assume a constant upstream pressure \( p_u \) and obtain solutions for the outgoing downstream pressure \( p_d^\pm \) based on incoming pressures \( p_d^- \). This amounts to finding the intersection of the flow curve and a straight line corresponding to Eq. (24), as illustrated in Fig. 11 below.

Figure 11: Flow and bore characteristic curves.

• However, it is preferable to maintain the dependence on \( p_\Delta \), which allows us to modify the mouth pressure in a time-domain simulation.

• Note that

\[
    p_d^+ - p_d^- = p_d^+ - p_d^- + [p_u - p_u] + [p_d^- - p_d^-] = p_u - 2p_d^- + p_u + [p_d^+ + p_d^-] = p_\Delta - p_\Delta,
\]

(25)

where \( p_\Delta = p_u - 2p_d^- \) consists of pressure components known or computable from previous values.

• We can then find a table of solutions, such as shown in Fig. 12 for the equation \( Z_c u_d(p_\Delta) = p_\Delta - p_\Delta \).

• For values of \( p_\Delta \) greater than \( p_c \), the reed is closed and the solutions are given by a straight line of slope one, which corresponds to zero flow and pressure reflection without inversion at the junction.

• Thus, when the reed is forced against the mouthpiece facing, the air column appears as a rigidly terminated or stopped end.
Figure 12: Solutions at reed/air column junction for input $p_{\Delta}^-$ ($p_c = 2280$ Pascals).

2.5 Reed Scattering Theory

• Alternately, the reed mechanism can be viewed as having a “lumped,” pressure dependent impedance given by:

$$Z_r(p_{\Delta}) = \frac{p_{\Delta}}{u(p_{\Delta})}.$$  \hspace{1cm} (26)

• Under the assumption that the reed motion is adequately described as “stiffness dominated” and in-phase with the driving pressure and that the resulting volume flow through the reed slit is also frequency-independent (ignoring the inertance of the air in the reed channel), this characterization can be evaluated in the time domain as a memory-less non-linearity.

• Viewing the reed/air column junction using scattering theory as seen from the air column side, pressure-wave reflection from an impedance of $Z_R$ is given by:

$$Z_r = \frac{P^+ + P^-}{(P^+ - P^-)/Z_c}.$$  \hspace{1cm} (27)

$$\frac{P^-}{P^+} = \frac{Z_r - Z_c}{Z_r + Z_c} = \frac{1 - Z_c/Z_r}{1 + Z_c/Z_r}.$$  \hspace{1cm} (28)

• The reed interface is then modeled with a nonlinear pressure-dependent reflection coefficient ($\rho(p_{\Delta})$) given by Eq. (28) and implemented via a scattering junction as shown in Fig. 13. The pressure entering the downstream instrument air column is determined as Smith, 1986:

$$p_d^+ = p_d^+ \cdot \rho(p_{\Delta}) + p_u^+ [1 - \rho(p_{\Delta})]$$

$$= p_u^+ - [p_u^+ - p_d^+] \rho(p_{\Delta}).$$  \hspace{1cm} (29)

• The reflection coefficient defined by the Bernoulli flow expression of Eq. (22) is shown in Fig. 14.

• For values of $p_{\Delta} > p_c$, the pressure reflection coefficient has unity gain. This corresponds to reflection from a rigidly stopped end, as mentioned above.

• A coefficient value of zero corresponds to a junction without discontinuity.

• A coefficient of -1.0 corresponds to an ideally open-pipe end (a reed mechanism of zero impedance).
For normal reed mechanisms with a narrow-slit geometry, it seems impossible that the reed/bore junction could ever be characterized by a reflection coefficient of zero or lower. It is expected that the impedance of the reed slit, even when unblown, has some minimum value (probably frequency-dependent, but we’ll ignore that for now) greater than $Z_c$.

### 2.6 The Reed as a Mass-Spring-Damper System

- While the normal playing range of clarinets and saxophones is typically well below the first resonance frequency of the reed, extended range techniques are becoming an increasingly common part of contemporary performance practice.
- In reality, the reed has some non-zero mass (the effective mass may vary with displacement and lip position) so that some phase delay will occur as the vibrating frequency of the reed increases. This behavior could have an important affect on the response of the instrument.
- The most common approach is to model the reed as a simple damped mechanical oscillator, as depicted in Fig. [15](#) with an equation of motion of the form

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx(t) = A_r \rho(p_\Delta(t)),$$

where $m$ is the equivalent reed mass, $k$ is the reed spring constant, and $b$ is the damping factor.
- A substantial portion of the damping comes from the player’s lower lip.
- Again, the reed motion is “displacement limited” by the mouthpiece facing.
Figure 15: The single-reed as a mechanical oscillator blown closed.

- The natural frequency of the system in the absence of damping and for constant reed parameters is \( \omega_r = \sqrt{k/m} \). Equation (30) is commonly expressed as

\[
\ddot{x}(t) + g_r \dot{x}(t) + \omega_r^2 x(t) = \frac{p_\Delta(t)}{\mu_r},
\]  

(31)

where \( g_r \) is the reed damping coefficient and \( \mu_r \) is the reed’s dynamic mass per unit area.

2.7 The Mass-Spring-Damper Solution

- As previously indicated, the flow through the reed channel is approximated “quasi-statically” using the Bernoulli equation and given by

\[
u = w (x + H) \left( \frac{2 |p_\Delta|}{\rho} \right)^{1/2} \text{sgn}(p_\Delta),
\]  

(32)

where \( w \) is the reed channel width, \( x \) is the time-varying reed position, calculated from Eq. (30), and \( H \) is the equilibrium tip opening.

- For single-reed geometries, the pressure and flow in the reed channel can be approximated as equivalent to the pressure and flow at the entrance to the instrument air column. This approximation is based on continuity and detachment of volume flow at the end of the reed channel such that pressure is not recovered in the mouthpiece.

- Thus, the acoustic interaction at the interface of the reed and air column can be solved using Eqs. (30) and (32), together with a description of the input impedance of the attached air column.

- For a cylindrical pipe, we can compute the impedance using the digital waveguide structure of Fig. 16.

\[
\delta[n] \rightarrow Z_c \rightarrow + \rightarrow z^{-M} \rightarrow R_L \rightarrow + \rightarrow p_0 \rightarrow p_x \rightarrow z^{-M} \rightarrow + \rightarrow \delta[n]
\]

Figure 16: A digital waveguide cylindrical pipe impedance model (from Scavone [1997]).

- From Fig. 16 it is clear that

\[
p_0 = 2p_0^- + Z_c u_0,
\]  

(33)

where \( p_0^- \) is the traveling-wave pressure entering the reed junction from the downstream air column.
Because of mutual dependencies, however, an explicit solution of these equations can be problematic. In a discrete-time computational context, these mutual dependencies can be understood to result in delay-free loops.

In [Guillemain et al. 2005], the reed system is discretized using a centered finite difference approximation that avoids a direct feedforward path through the reed transfer function. The resulting system equations can then be expressed in terms of a second-order polynomial equation and an explicit solution found.

The centered finite-difference approximation of Eq. (30) results in a digital filter structure of the form

\[ \frac{X(z)}{P_\Delta(z)} = \frac{-z^{-1}/\mu_r}{(f_r^2 + \frac{g_r}{2}) + (\omega_r^2 - 2f_r^2)z^{-1} + (f_s^2 - \frac{2g_s}{\omega_s^2})z^{-2}}, \]  

(34)

where \( f_s \) is the computational sample rate.

As noted in [Guillemain 2004], however, this filter structure is only stable for \( \omega_r < f_s \sqrt{4 - (g_r/\omega_s)^2} \), limiting its use at low sample rates and/or with high reed resonance frequencies.

A direct application of the bilinear transform to the system of Eq. (30) results in a digital filter structure given by

\[ \frac{X(z)}{P_\Delta(z)} = \frac{-1/\mu_r [1 + 2z^{-1} + z^{-2}]}{a_0 + 2(\omega_r^2 - \alpha^2)z^{-1} + (\alpha^2 - g_r \alpha + \omega_r^2)z^{-2}}, \]  

(35)

where \( a_0 = \alpha^2 + g_r \alpha + \omega_r^2 \) and \( \alpha \) is the bilinear transform constant that controls frequency warping.

Note that we can achieve an exact continuous- to discrete-time frequency match at the resonance frequency of the reed by setting \( \alpha = \omega_r / \tan(\omega_r/2f_s) \).

In this case, the use of the bilinear transform guarantees a stable digital filter at any sample rate. The presence of the direct feedforward path in Eq. (35), however, prohibits the explicit reed interface solution mentioned above.

We therefore seek an alternative form of Eq. (35) that preserves stability and avoids an undelayed feedforward coefficient in the transfer function numerator.

By default, the bilinear transform substitution produces a system with “zeroes” at \( z = \pm 1 \) (or at frequencies of 0 and \( f_s/2 \) Hz). While this result is often desirable for digital resonators, we can modify the numerator terms without affecting the essential behavior and stability of the resonator.

In fact, it is the numerator terms that control the phase offset of the decaying oscillation. Thus, we can modify and renormalize the numerator to produce a filter structure of the form

\[ \frac{X(z)}{P_\Delta(z)} = \frac{-4z^{-1}/\mu_r}{a_0 + 2(\omega_r^2 - \alpha^2)z^{-1} + (\alpha^2 - g_r \alpha + \omega_r^2)z^{-2}}. \]  

(36)

The frequency- and time-domain responses of the centered finite-difference and “modified” bilinear transform filter structures are shown in Fig. 17 for a reed resonance frequency \( f_r = 2500 \) Hz and \( f_s = 22050 \) Hz.

The complete clarinet model involves the calculation of the reed displacement using this stable reed model, the volume flow through the reed channel as given by Eq. (32), and the relationship between flow and pressure at the entrance to the air column as given by Eq. (33).

Because the reed displacement given by Eq. (30) does not have an immediate dependence on \( p_\Delta \), it is possible to explicitly solve Eqs. (33) and (22), as noted in [Guillemain et al. 2005], by an expression of the form

\[ u_0 = 0.5 \left( B \sqrt{(Z_c B)^2 + 4A} - Z_c B^2 \right) \text{sgn}(A), \]  

(37)

where \( A = p_m - 2p_0 \) and \( B = w(x + H)(2/\rho)^{1/2} \) can be determined at the beginning of each iteration from constant and past known values.
• Whenever the reed channel height \( x + H < 0 \), \( u_0 \) is set to zero and \( p_0 = 2p_0 \).

• In Fig. 18, the normalized pressure response of the complete DW synthesis model is plotted using both reed models with \( f_r = 2500 \) Hz and \( f_s = 22050 \) Hz.

• The behaviors are indistinguishable for these system parameters, though as indicated above it is possible to run the modified bilinear transform model at significantly lower sample rates (and with higher reed resonance frequencies).

2.8 Refinements

Much of the discussion thus far has been based on rather ideal behaviors and/or system properties. Despite that, these analyses provide good “first-order” descriptions for the various components of the reed mechanism and together, capture the fundamental response of the overall system. Below, we discuss a number of incremental refinements that have been proposed to these equations.
• The motion of the reed is expected to induce a flow contribution of its own proportional to $A_r \dot{x}(t)$.

• The flow that enters the reed channel from the upstream side is therefore considered to divide into a component $u_d$ entering the downstream air column and the volume flow swept out by the reed,

$$u(t) = u_d(t) + A_r \dot{x}(t).$$  \hspace{1cm} (38)

• This extra flow component has been shown to have some significant influence on playing frequencies as predicted from analytical and numerical simulations \cite{Coyle2015}.

• \cite{Stewart1980} and \cite{Sommerfeldt1988} modeled the reed as a damped, driven, nonuniform bar using a fourth-order differential equation. In this way, changes in effective mass and stiffness with bending along the curvature of the lay are automatically incorporated.

• \cite{Avanzini2004} also modeled the reed as a clamped-free bar using finite difference techniques and derived appropriate parameters from measurements and tuning. These results were then used to develop a non-linear lumped model \cite{vanWalstijn2007}.

• More recent models of reed stiffness include an extra force term which only becomes active when the reed displacement goes beyond a certain distance, thus effectively producing a non-linear spring effect \cite{Chatziioannou2012}.

• An additional force on the reed caused by higher localized volume flow within the reed channel was proposed by \cite{Worman1971} and had been included in a few later reed analyses \cite{Benade1976, Schumacher1981, Keefe1990}. For such a force to exist, the reed channel height must be non-uniform or the flow must separate at the channel entrance and then subsequently reattach at a further point in the channel. \cite{Hirschberg1990} point out that there are incompatibilities in the derivation of the “Bernoulli” force that make its application to the clarinet reed mechanism questionable.

2.9 The Brass Lip Mechanism

• The brass lip mechanism functions as a pressure-controlled valve that admits a puff of air whenever the pressure is high in the mouthpiece.

• Because the resonance frequency of the brass lip mechanism must nearly match the desired sounding frequency of the instrument (hopefully with support from an air column resonance at or near that frequency), the mass component of the lips cannot be ignored.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{brass_lip_mechanism.png}
\caption{The brass player’s lips as a mechanical oscillator blown open.}
\end{figure}

• Pressure pulses reflected back from the far end of the horn tend to force the player’s lips open ... positive feedback.
• While one might expect the lips to be blown closed when the mouthpiece pressure is high, it should be remembered that the mass in a mass-spring-damper system lags behind the driving force by 1/4 cycle at its resonance frequency (as shown in Fig. 20). During this “lag time”, the lip valve is open and flow through the valve can augment the positive mouthpiece pressure.

• The player controls the resonance frequency of his/her lips via tension and mass (position) variations.

• Oscillations are favored when the air column has one or more resonances that correspond to the harmonics of the fundamental pitch.

• Most of the equations discussed earlier for the woodwind reed valve can also be applied to the brass lip mechanism. The primary distinction concerns the equations of lip displacement. The lips are initially closed but then blown open.

• Attempts to model the lip mechanism have included both “swinging-door” and “up-down” displacement trajectories.

3 Modeling Vocal Tract Influence

In this section, we explore the influence of upstream vocal tract resonances in reed wind instrument performance and modeling. Vocal tract manipulations are a common, though sometimes subtle, performance practice exploited by experienced musicians to produce a variety of important acoustic effects, including contemporary performance techniques such as multiphonics and extended range playing. Several previous acoustic studies have been conducted and most agree that the upstream system can have significant influence under certain circumstances. There is less agreement regarding the importance of this mechanism in “traditional” playing ranges and conditions.

3.1 Background

• The 1980s were an active period for acoustic investigation into the role and influence of a player’s vocal tract in wind instrument performance Clinch et al. [1982], Benade and Hoekje [1982], Backus [1985], Benade [1985], Hoekje [1986].

• While a few of these reports included suspect conclusions or demonstrated unfamiliarity with advanced performance practice techniques, a high level of understanding was achieved by the end of the decade.
Clinch et. al. [1982] performed X-ray fluoroscopic examinations of vocal tract shape changes involved in the playing of the clarinet, soprano saxophone, and recorder. They noted a strong dependence of note quality on vocal tract shape and, somewhat curiously, concluded “that vocal tract resonant frequencies must match the frequency of the required notes in clarinet and saxophone performance.”

Backus [1985] made vocal tract impedance measurements and found peak values an order of magnitude less than the impedances of the clarinet air column resonances.

Backus also experimented with a clarinet-like system arranged to sound using a vacuum mechanism located at its downstream end. He observed relatively little change in the resulting waveforms when either human or more sharply tuned resonance structures were placed around the vibrating reed/mouthpiece system.

From these results, Backus concluded that “the player’s vocal tract has a negligible influence on the instrument tone.”

By assuming continuity of volume flow, Benade and Hoekje [Benade and Hoekje 1982, Benade 1985, Hoekje 1986] showed that the pressure and flow relationships on each side of the reed can be written as

\[ U = \frac{P_d}{Z_d} + \frac{P_d - P_u}{Z_r} \quad - U = \frac{P_u}{Z_u} + \frac{P_u - P_d}{Z_r}, \]  

(39)

where \( Z_u \) is the input impedance looking upstream from the reed into the player’s windway, \( Z_d \) is the input impedance looking downstream from the reed into the instrument air column, and \( Z_r \) is the nonlinear acoustic impedance of the reed valve.

The flow through the reed aperture can then be expressed in terms of the pressure difference \( P_\Delta = P_u - P_d \) and the above equations solved as \( -P_\Delta = ZU \) where

\[ Z = \frac{Z_r(Z_d + Z_u)}{Z_r + Z_d + Z_u} = Z_r \parallel (Z_d + Z_u). \]  

(40)

The reed impedance plays a secondary role in this expression because it tends to be very large in comparison to the other impedances.

If the upstream impedance \( (Z_u) \) is negligible, as was assumed for many years, the system can be accurately described in terms of the air column and reed impedances alone. On the other hand, it is clear that if significant impedance peaks occur in the upstream system, they can influence the behavior of the instrument in important ways.

Benade and Hoekje made upstream impedance measurements and found that certain vocal tract configurations can produce strong upstream impedance peaks.

In addition, they noted a number of ways in which upstream resonances could have considerable influence on the entrainment of the reed and the resulting sound spectra.

With respect to the lack of earlier recognition within the acoustics community of the possible influences of a player’s windway, Benade [1985] noted that:

1. every player quickly learns to avoid windway configurations that might adversely affect the instrument response and/or produce undesirable multiphonics;
2. the audible effects of resonance alignment in the player’s windway are rather subtle and not easily recognized in the resulting instrument spectrum;
3. the ability to make use of vocal tract resonances to strengthen or support instrument oscillations is a refinement that typically comes only with many years of performance experience.

In a later study by Wilson [1996], upstream resonances were examined during clarinet performance of several musical phenomena.
• She found that the performer tends to align upstream resonances with the first or second harmonic of a sounding tone, “but that there were also a number of tones that did not have an airway resonance aligned with a harmonic.”

• For pitchbend, a large-amplitude vocal tract resonance was used to control the playing frequency. When playing multiphonics, Wilson found that the performer creates a resonance that supports an oscillation at a linear combination of the audible pitch frequencies.

• Sommerfeldt and Strong [1988] presented a detailed time-domain simulation of a player-clarinet system that included a sixteen segment cylindrical tube approximation for the player’s windway.

• They explored several vocal tract configurations and found some instances of upstream influence on the resulting sound spectra.

3.2 The Instrument Air Column

• The instrument air column is modeled as a single uniform waveguide of either cylindrical or conical shape and appropriately designed scattering junctions are applied at each of its ends.

• While more intricate air column structures can be modeled with DW techniques [Välimäki 1995, Scavone 1997, Van Walstijn 2002], such additional complexity is unnecessary for the purposes of this study.

• The block diagram shown in Fig. 21 models traveling-wave propagation within a uniform air column structure. The single digital filter $R(z)$ accounts for the combined frequency-dependent losses attributable to radiation, thermal heat conduction, and viscosity along the air column walls.

![Figure 21: Generalized digital waveguide air column structure.](image)

3.3 The Reed Junction

• The single-reed woodwind excitation mechanism can be reasonably well modeled as a nonlinear spring because it is normally driven well below its resonance frequency.

• Using DW techniques, this characteristic is transformed into a nonlinear reflection function and implemented via a scattering junction as shown in Fig. 22. The pressure entering the downstream instrument air column is determined as:

\[
p_d^+ = p_d^- \cdot \rho(p_\Delta) + p_u^+ [1 - \rho(p_\Delta)]
\]

\[
= p_u^+ - [p_u^+ - p_d^-] \rho(p_\Delta),
\]

where $\rho(p_\Delta)$ is the nonlinear reed reflection coefficient.

• Details regarding the derivation of $\rho(p_\Delta)$ in the context of a traveling-wave, scattering theory approach are available elsewhere [Smith 1986, Scavone 1997].

• Pressure scattering on the upstream side of the reed junction is given by

\[
p_u^- = p_d^- - [p_u^+ - p_d^-] \rho(p_\Delta).
\]
3.4 Upstream Windway Models

- With a few exceptions, most wind instrument simulations have assumed a constant or slowly varying pressure in the player's mouth and otherwise ignored possible upstream influences.

- Under these assumptions, the upstream system can be considered a large reservoir driven by a zero-frequency (DC) current source.

- An electrical circuit analog for such a system is shown in Fig. 22. The current source $U_l$ represents the player's lungs, while flow resistance in the lungs and trachea is characterized by $R_l$.

- In general, the lung impedance varies over time based on the vocal fold configuration.

- The cavity impedance is given by $Z_c = -j\rho c^2/(V\omega)$, where $\rho$ is the mass density of air, $c$ is the speed of sound in air, $V$ is the volume of the cavity, and $\omega$ is the radian frequency.

- The upstream resistance parameter $R_u$ characterizes losses in the player’s windway.

- The impedance seen by the reed looking upstream is infinite for steady flow but relatively small at higher frequencies.

- Under these conditions, the reed is controlled by the oscillating pressure on its downstream side and the DC upstream pressure only.

3.5 Windway Resonances

- The primary goal of this study was to investigate how upstream resonances might influence the resulting instrument sound and the oscillations of the reed valve.

- With this in mind, simple vocal tract characterizations having control parameters directly tied to resonance peak and bandwidth features were explored.

- Note that this is not equivalent to modeling the mouth cavity as a Helmholtz resonator, which has an impedance minimum at resonance.

- An upstream system with a single resonance is represented by the electrical circuit analog of Fig. 24.

- The impedance seen from the reed is characterized by peaks at DC (set with $C_v$) and at the resonance frequency, which is determined by the components $L_1$, $C_1$, and $R_1$. 

Figure 22: The reed scattering junction.

Figure 23: Electrical circuit analog for a traditional upstream windway system.
Figure 24: Electrical circuit analog for upstream windway with a single resonance.

- Despite the extreme simplicity of this characterization, wind instrument performers are typically making use of just a single resonance in their windway to influence the response of the reed.

- Within the digital waveguide context, the lumped impedance representation of the upstream system is converted to a traveling-wave scattering junction expressed in terms of reflectances and transmittances.

- Figure 25 shows a representative reflectance characteristic when the lung and trachea impedance is assumed infinite.

![Figure 25: Upstream reflectance derived from the circuit of Fig. 24.](image)

- The complete system of Fig. 24 can be transformed to a traveling-wave scattering characterization using a transmission-matrix approach.

- If the series combination of the resonant circuit and volume capacitance are represented by an impedance $Z_s$ and the upstream resistance by $Z_a = R_u$, the following matrix approach can be followed:

$$
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
Z_s^{-1} & 1
\end{bmatrix} \begin{bmatrix}
1 & Z_a \\
0 & 1
\end{bmatrix} \begin{bmatrix}
P_2 \\
U_2
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & Z_a \\
Z_s^{-1} & 1 + \frac{Z_a}{Z_s}
\end{bmatrix} \begin{bmatrix}
P_2 \\
U_2
\end{bmatrix},
$$

(43)

- To render these relationships in the digital waveguide domain, it is necessary to transform the plane-wave physical variables of pressure and volume velocity to traveling-wave variables as

$$
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
P_1^+ + P_1^- \\
Z_0^{-1} \left( P_1^+ - P_1^- \right)
\end{bmatrix},
$$

(44)

where $Z_0$ is the characteristic impedance of the section.

- Waveguide pressure variables on both sides of the upstream system are then related by an expression of the form

$$
\begin{bmatrix}
P_1^- \\
P_2^+
\end{bmatrix} = \begin{bmatrix}
\mathcal{R}^- & \mathcal{T}^- \\
\mathcal{T}^+ & \mathcal{R}^+
\end{bmatrix} \begin{bmatrix}
P_1^+ \\
P_2^-
\end{bmatrix}.
$$

(45)
The process of deriving appropriate discrete-time reflectance and transmittance filters is detailed elsewhere with respect to woodwind tonehole modeling [Scavone 1997].

For the system of Fig. 24, the resulting implementation requires four third-order digital filters.

A simplified, intuitive approach is illustrated by the block diagram of Fig. 26.

\[
\begin{align*}
    p_t^+ & \xrightarrow{+} p_u^+ \\
    g & \xrightarrow{} \sum \\
    z^{-1} & \xrightarrow{} p_u^-
\end{align*}
\]

Figure 26: A simplified upstream resonance block diagram.

A single second-order digital resonator is used to model the upstream resonance while the lung pressure component of the model is extracted and simply added to the reflected upstream pressure component.

A coupling constant \( g \) is included to control the relative level of upstream influence.

The unit delay shown in this signal path is necessary to avoid a delay-free loop through the digital resonance filter and reed scattering junction.

From this structure, it should be obvious that second-order digital resonators can be cascaded in parallel to simulate multiple upstream resonances.

However, because vocal tract resonances will not typically have harmonic relationships, it is unlikely that a performer would be able to manipulate the upstream system in such a way that multiple upstream resonances could be used to reinforce multiple downstream resonances.

### 3.6 Piecewise Cylindrical Approximations

A distributed acoustic model of the vocal tract can be developed by approximating the dimensions of the upstream windway with a series of concatenated cylindrical pipe sections.

In the digital waveguide context, each cylindrical section is efficiently implemented with a single digital delay-line and a one-multiply scattering junction.

This approach was previously used to create an articulatory vocal tract model for the synthesis of singing [Cook 1990].

With a model capable of accurately simulating arbitrary vocal tract profiles, it is possible to explore general windway shape trends and influences as reported by Clinch et al. [Clinch et al. 1982].

A multi-segment cylindrical model of the vocal tract was implemented for this study, though its use presented several challenges.

In general, it is difficult to predict the way changes in vocal tract shape will affect the resonance structure of the upstream system.

Further, the resulting parameter space is complex and requires a well developed, intuitive control interface.

Finally, such complexity is unnecessary when one considers that the performer typically makes use of just a single windway resonance to influence the vibrations of the reed.
References


